Edge-exchangeable graphs, sparsity, and power laws

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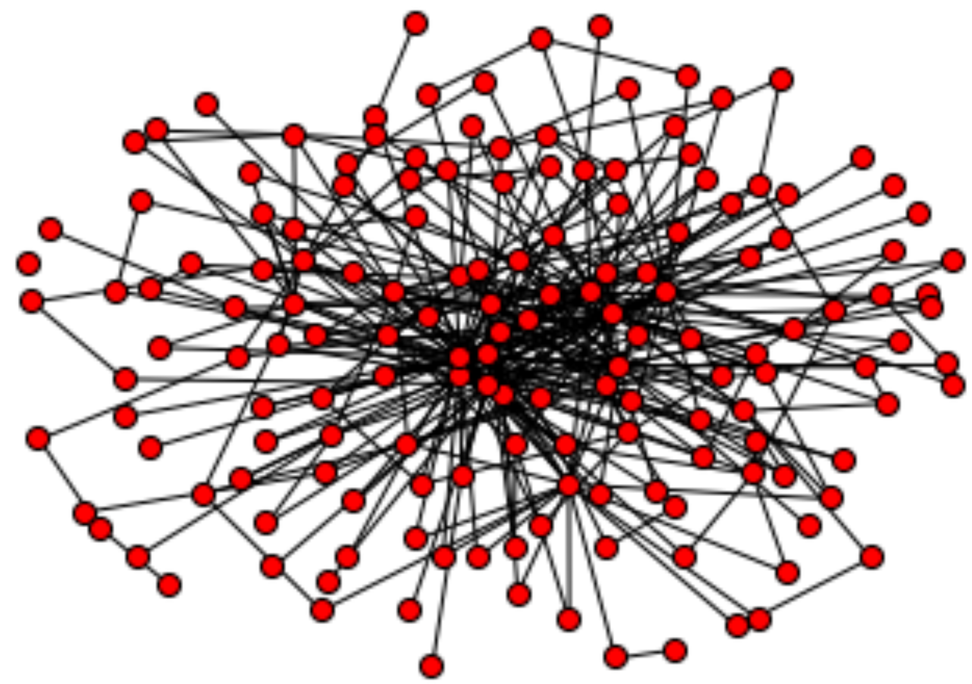
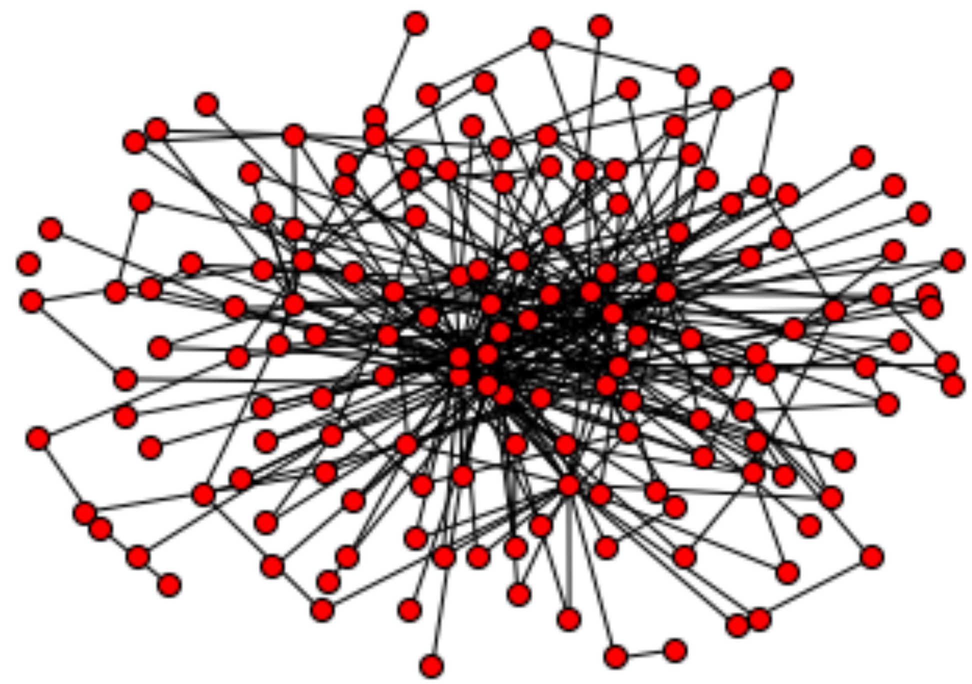
Assistant Professor,
MIT

Joint work with: Diana Cai, Trevor Campbell

Probabilistic models for graphs



Probabilistic models for graphs

$p($  $)$

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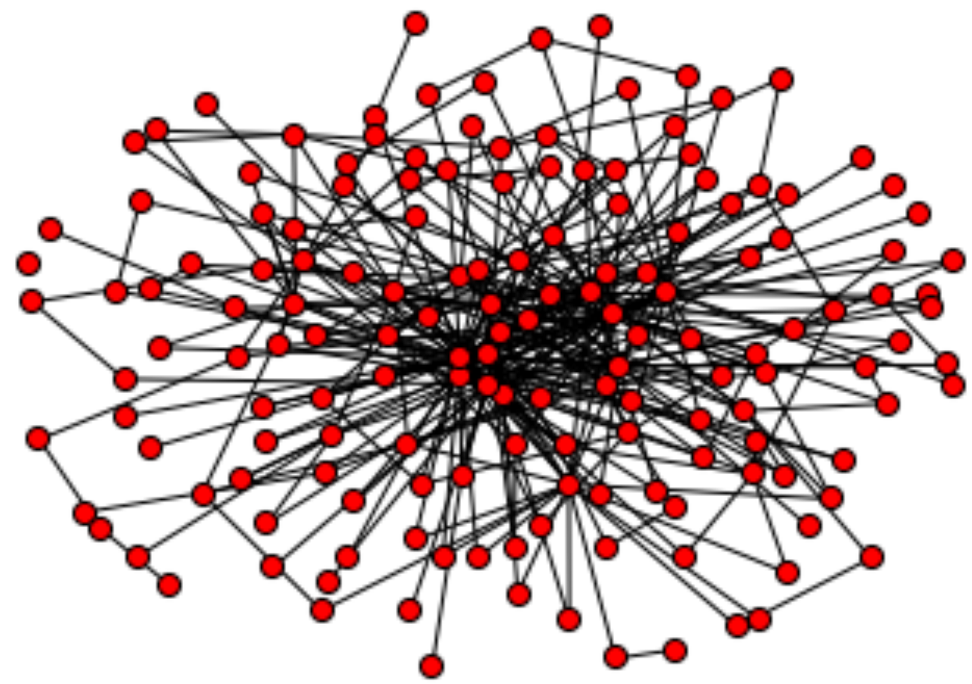
$p($  $)$

- Stochastic block model, mixed membership stochastic block model, infinite relational model, etc

[Holland et al 1983, Kemp et al 2006, Xu et al 2007, Airoldi et al 2008]

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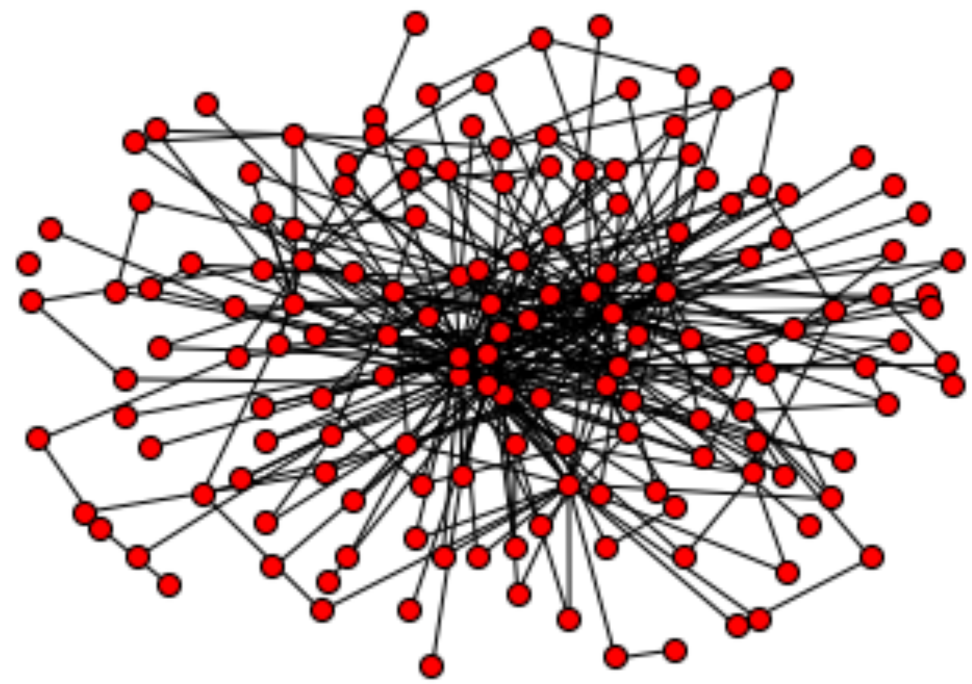
$p($  $)$

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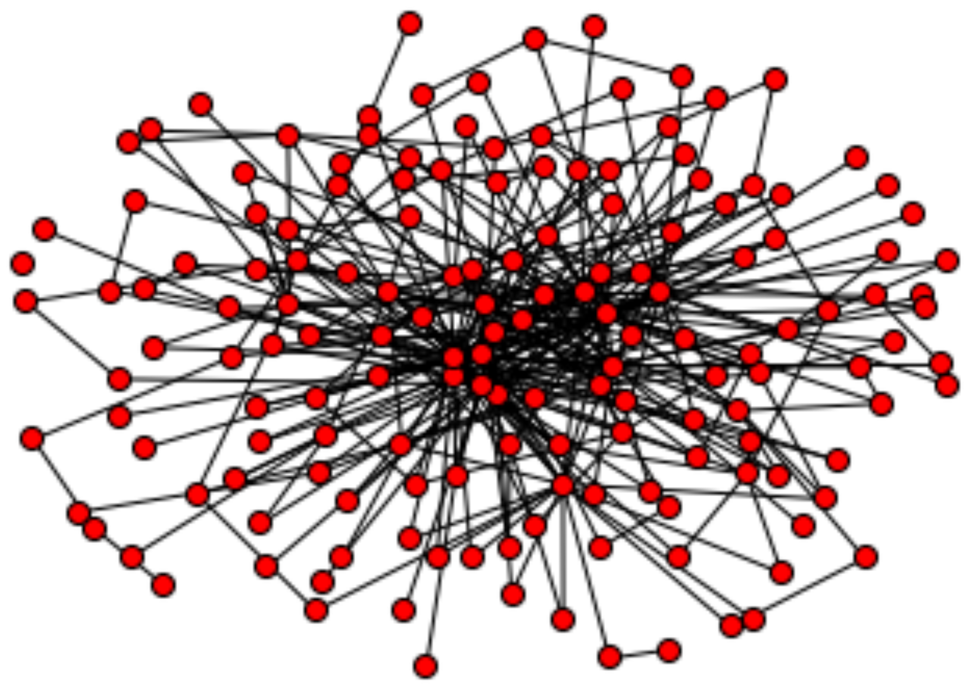
$p($  $)$

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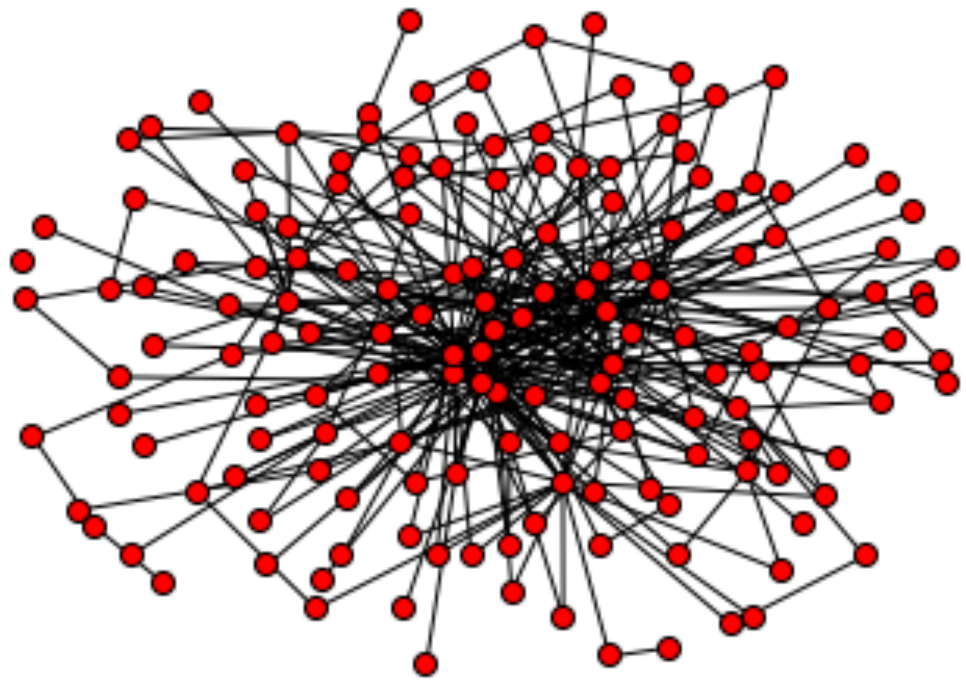

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- Stochastic block model, mixed membership stochastic block model, infinite relational model, etc
- Learn hidden structure, uncertainties, modularity
- Problem: model misspecification, dense graphs
- Solution: a new set of models for sparse graphs

[Holland et al 1983, Kemp et al 2006, Xu et al 2007, Airoldi et al 2008]

De Finetti

- A data sequence $X_1, X_2, \ldots$

[Hewitt, Savage 1955; Aldous 1983]

De Finetti

- A data sequence is *infinitely exchangeable* if the distribution of any *N* data points doesn't change when permuted: $p(X_1, \ldots, X_N) = p(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$

[Hewitt, Savage 1955; Aldous 1983]

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- *De Finetti's Theorem* (roughly): A sequence $X_1, X_2, \ldots$ is infinitely exchangeable if and only if, for all *N* and some distribution *P*:

$$p(X_1, \ldots, X_N) = \int_\theta \prod_{n=1}^{N} p(X_n|\theta) P(d\theta)$$

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- Motivates:

[Hewitt, Savage 1955; Aldous 1983]

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- Motivates:
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- Motivates:
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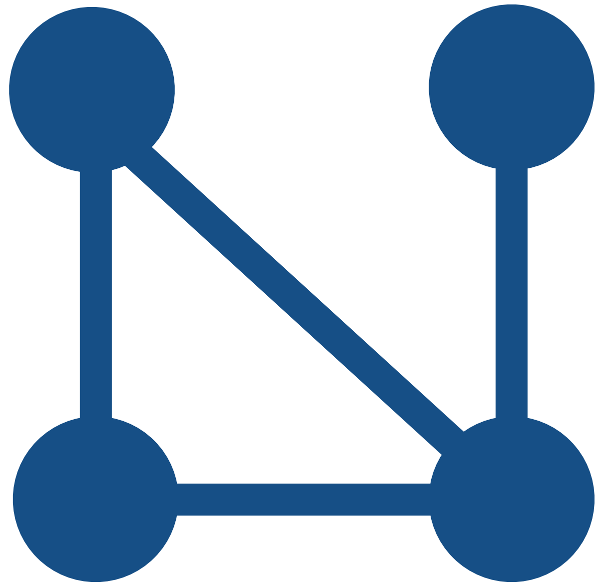
    - *Nonparametric Bayesian* priors

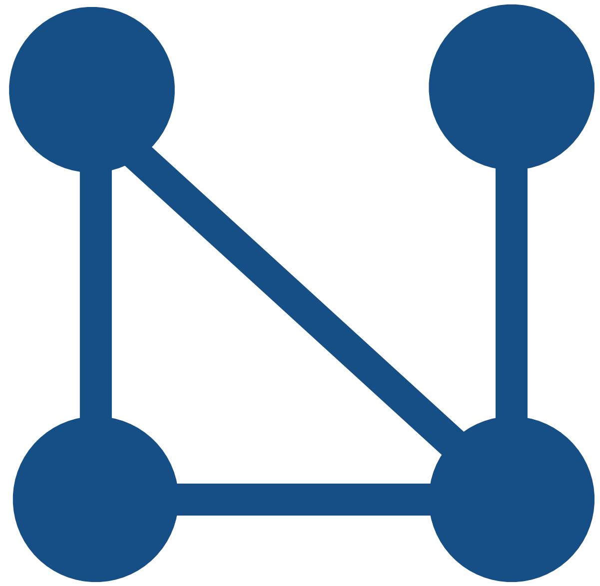
[Hewitt, Savage 1955; Aldous 1983]

Sequence of graphs



G

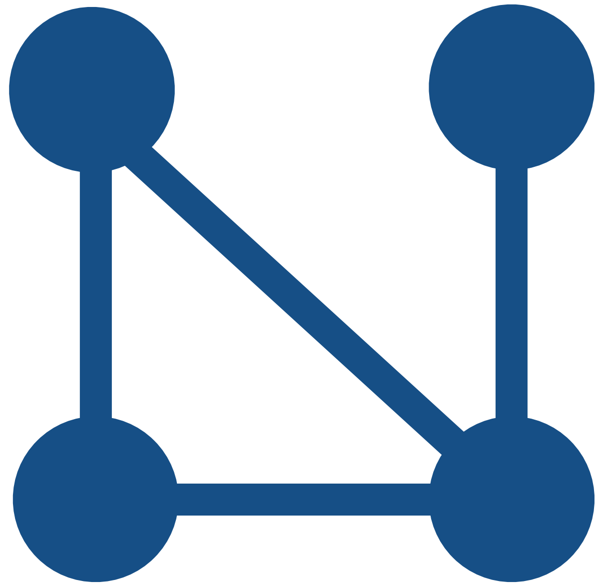
Sequence of graphs



G

- Nodes/vertices

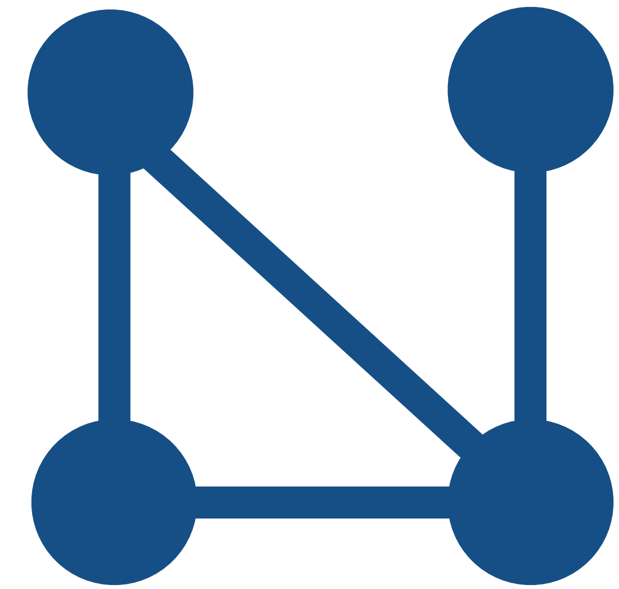
Sequence of graphs



G

- Nodes/vertices
- Edges

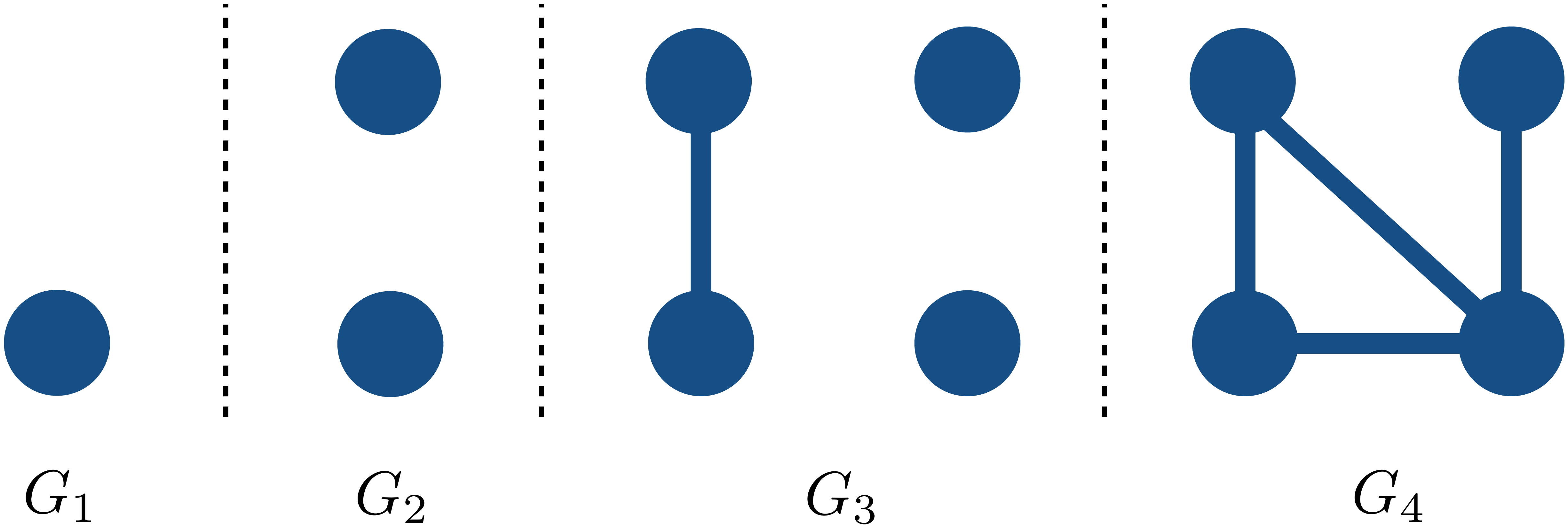
Sequence of graphs



G

- Nodes/vertices
- Edges
- Active nodes

Sequence of graphs



- Nodes/vertices
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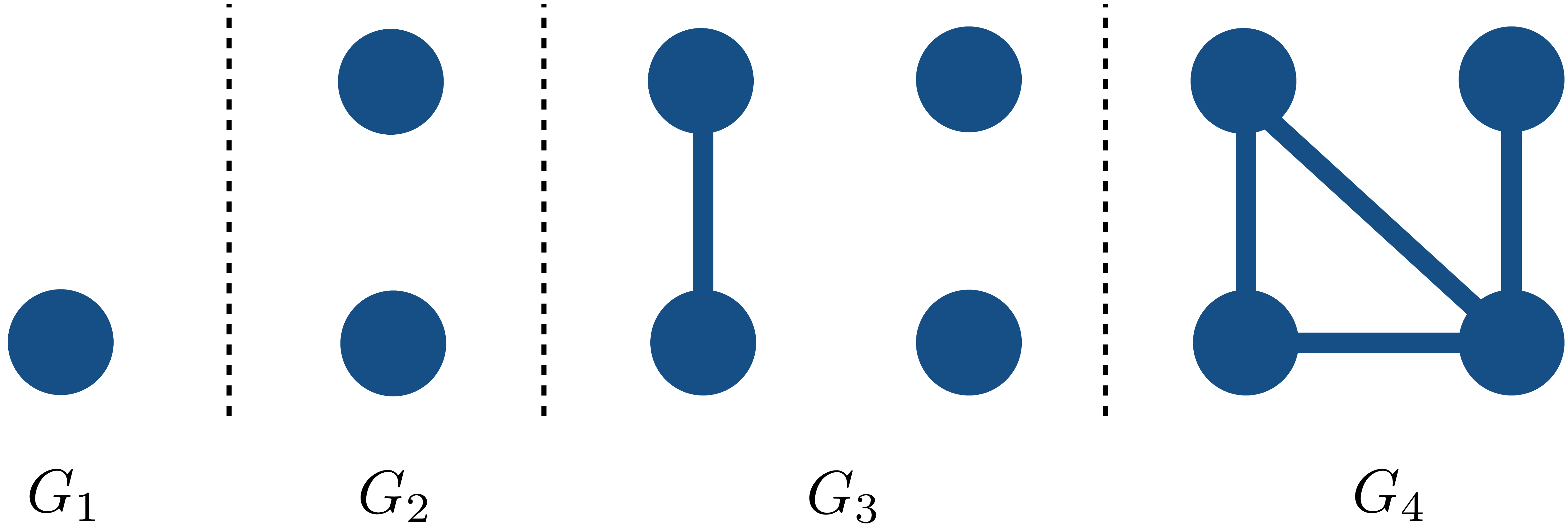

- Nodes/vertices
- Edges
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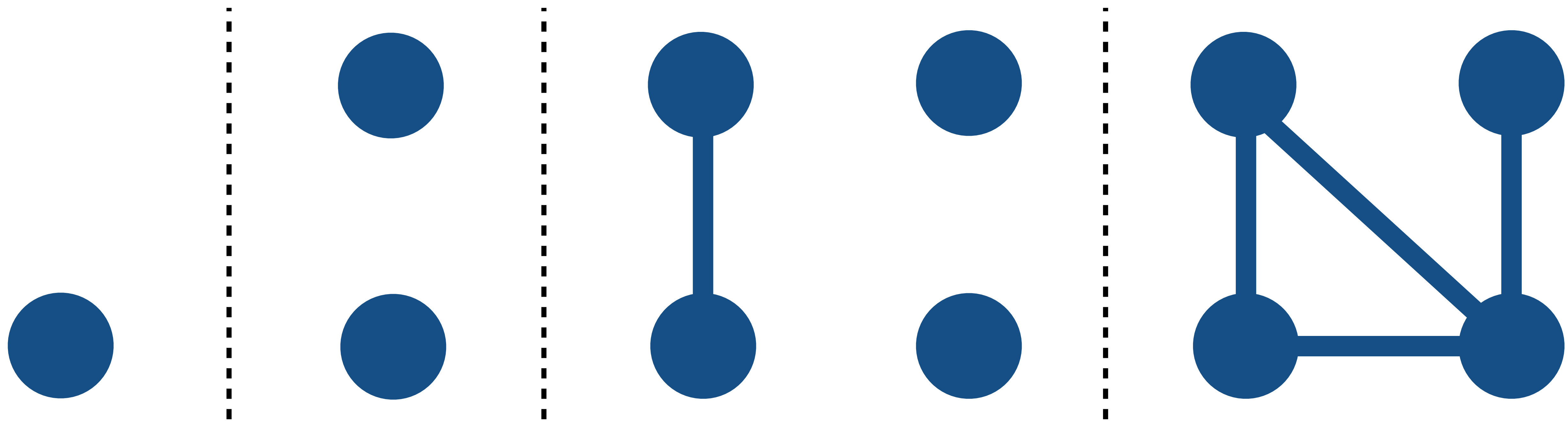
- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$

Sequence of graphs

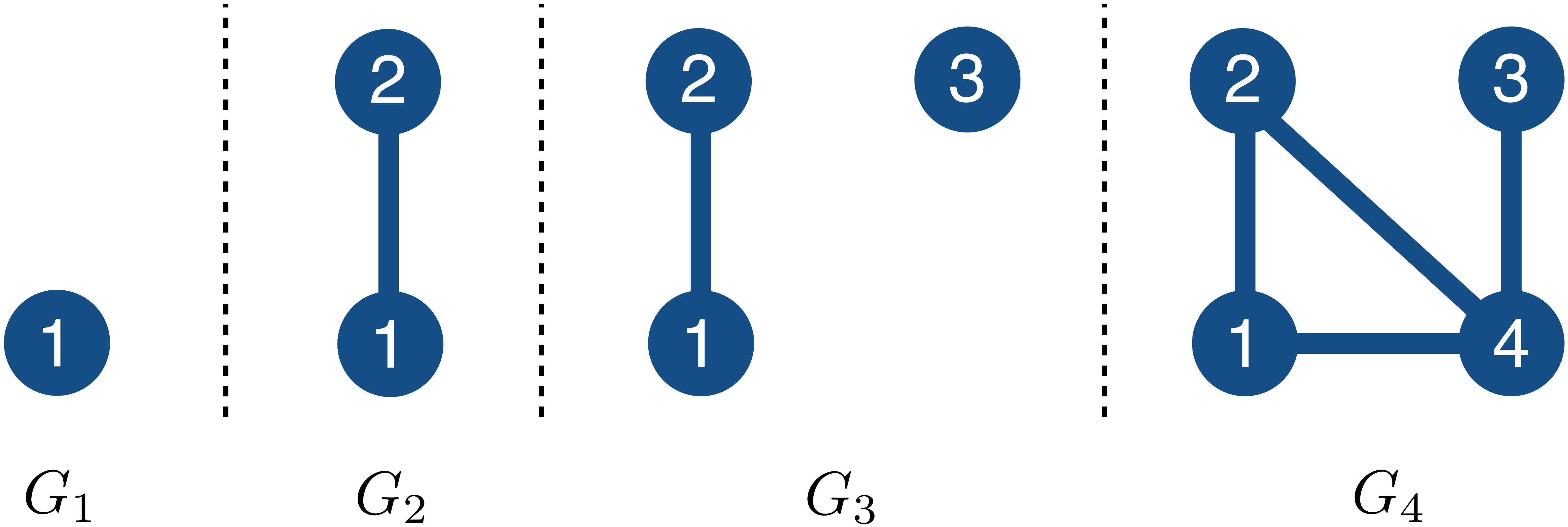


G_1 G_2 G_3 G_4

- Nodes/vertices
- Edges
- Active nodes

- *Dense* graph sequence $\#\text{edges}(G_n) \geq c \cdot [\#\text{nodes}(G_n)]^2$
- *Sparse* graph sequence $\#\text{edges}(G_n) \in o([\#\text{nodes}(G_n)]^2)$

(Node) Exchangeability



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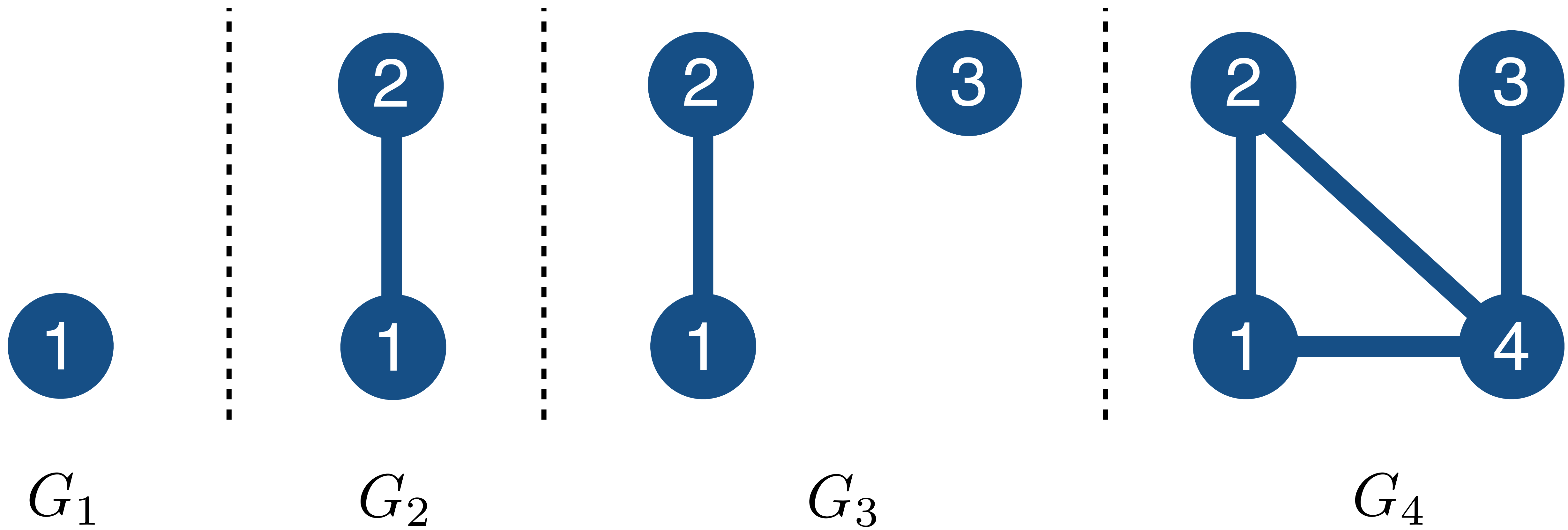
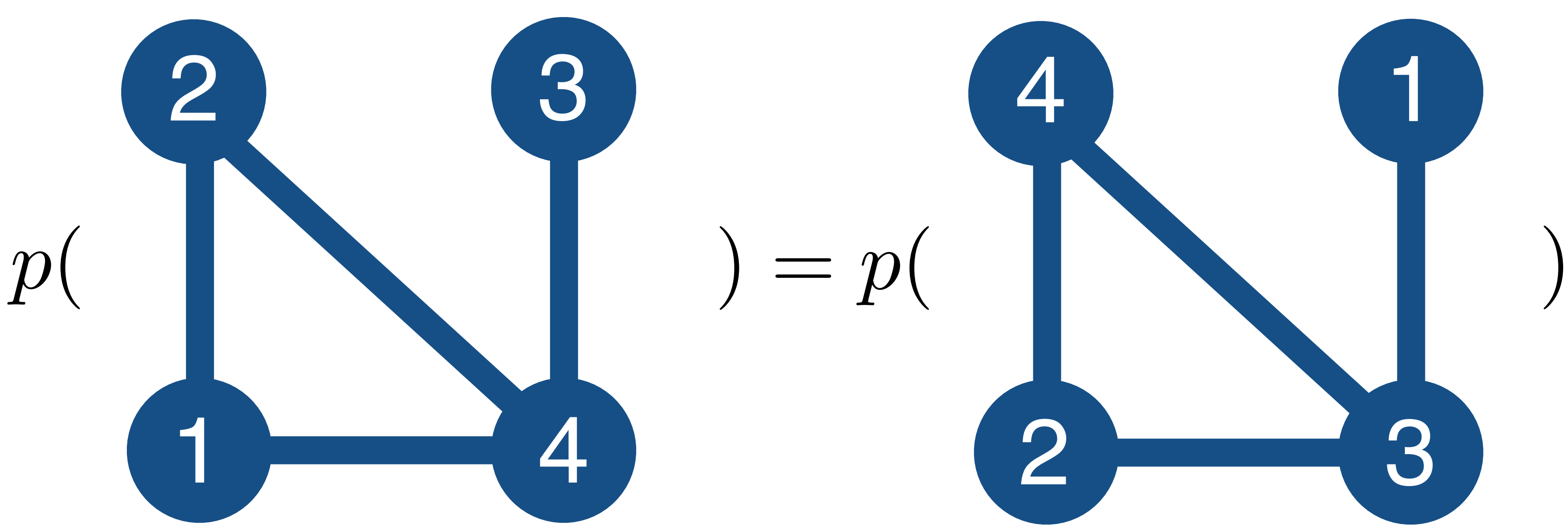
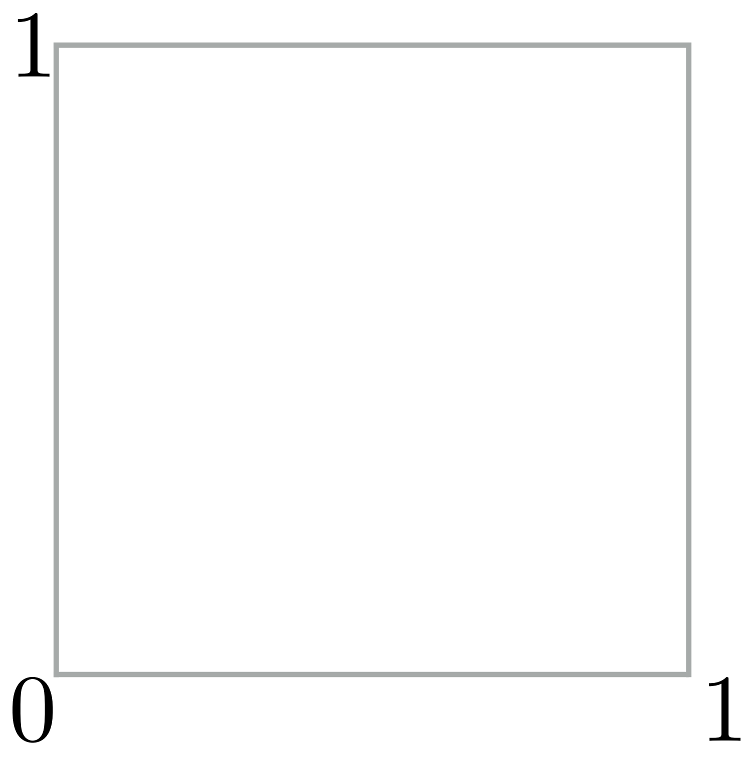




Aldous-Hoover



[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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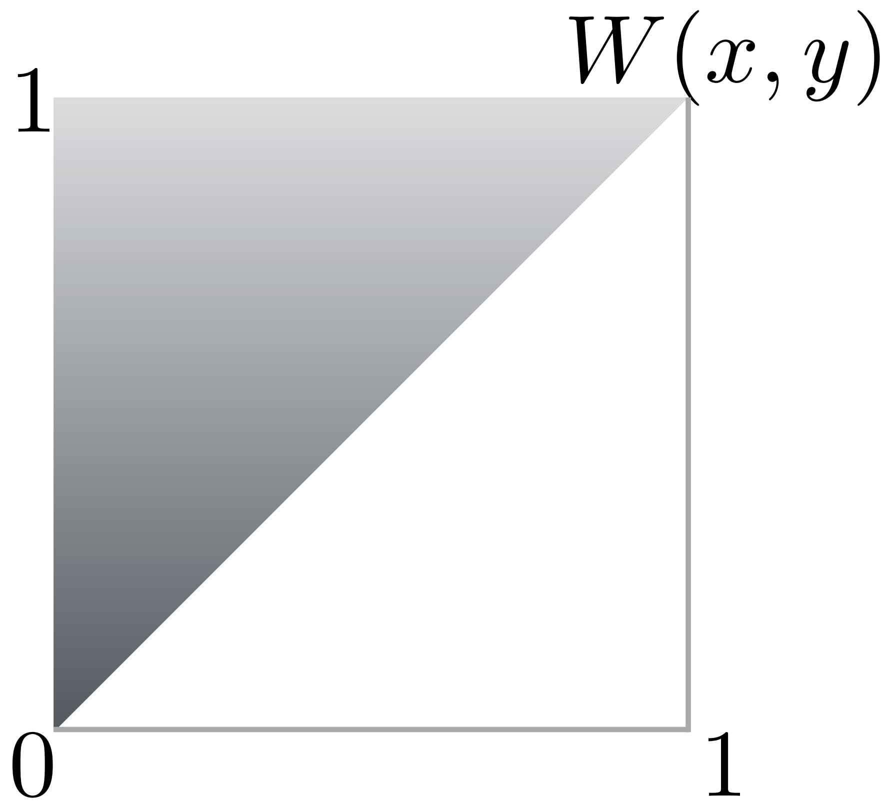

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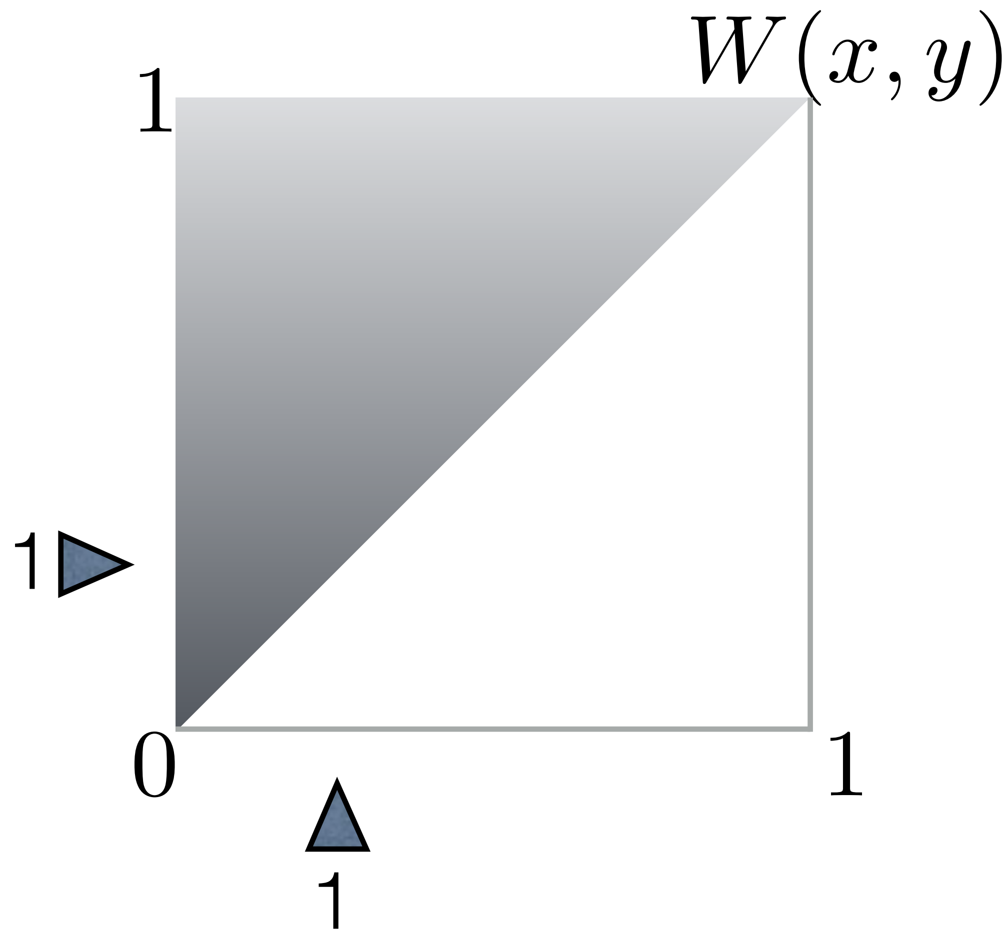

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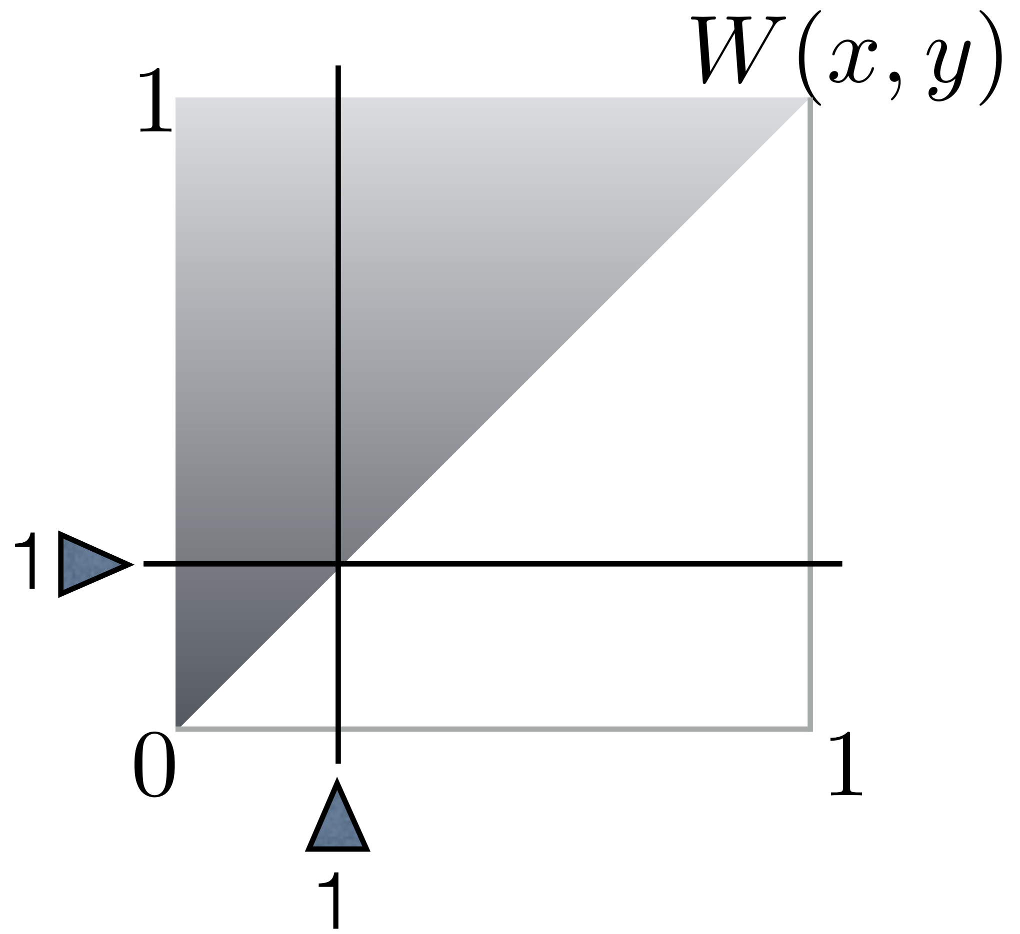

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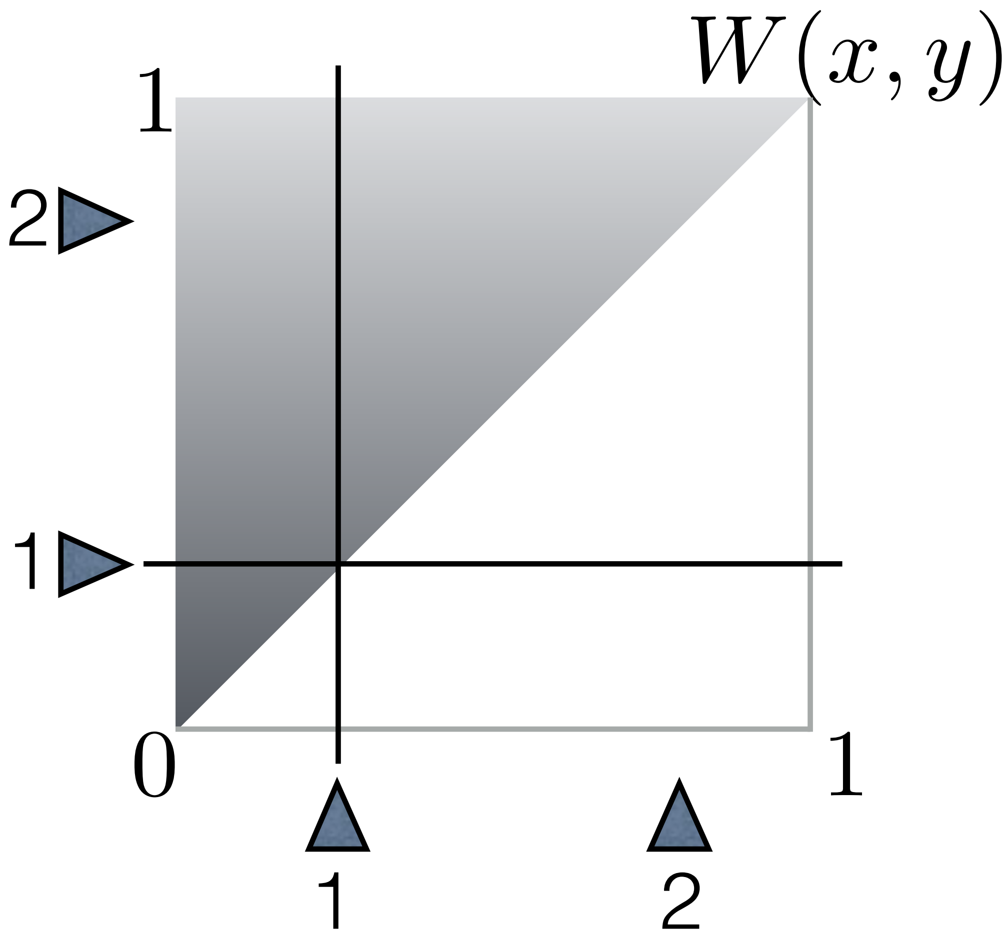

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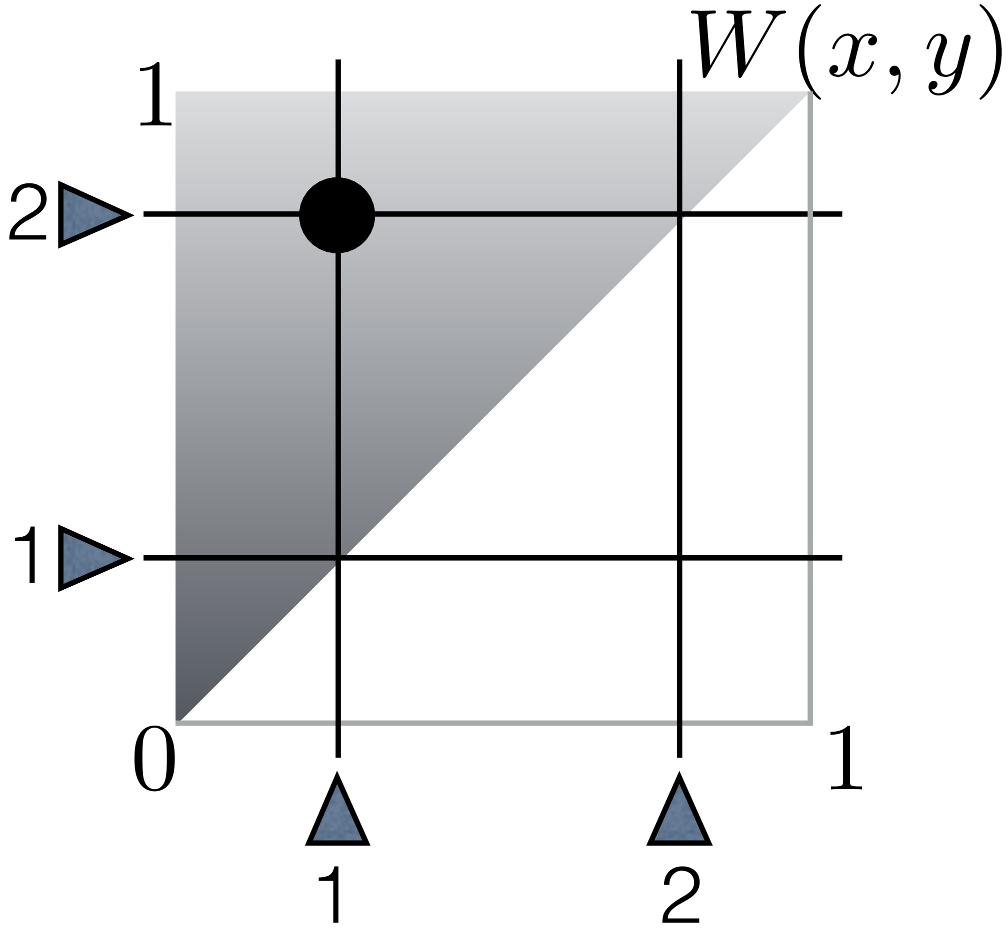

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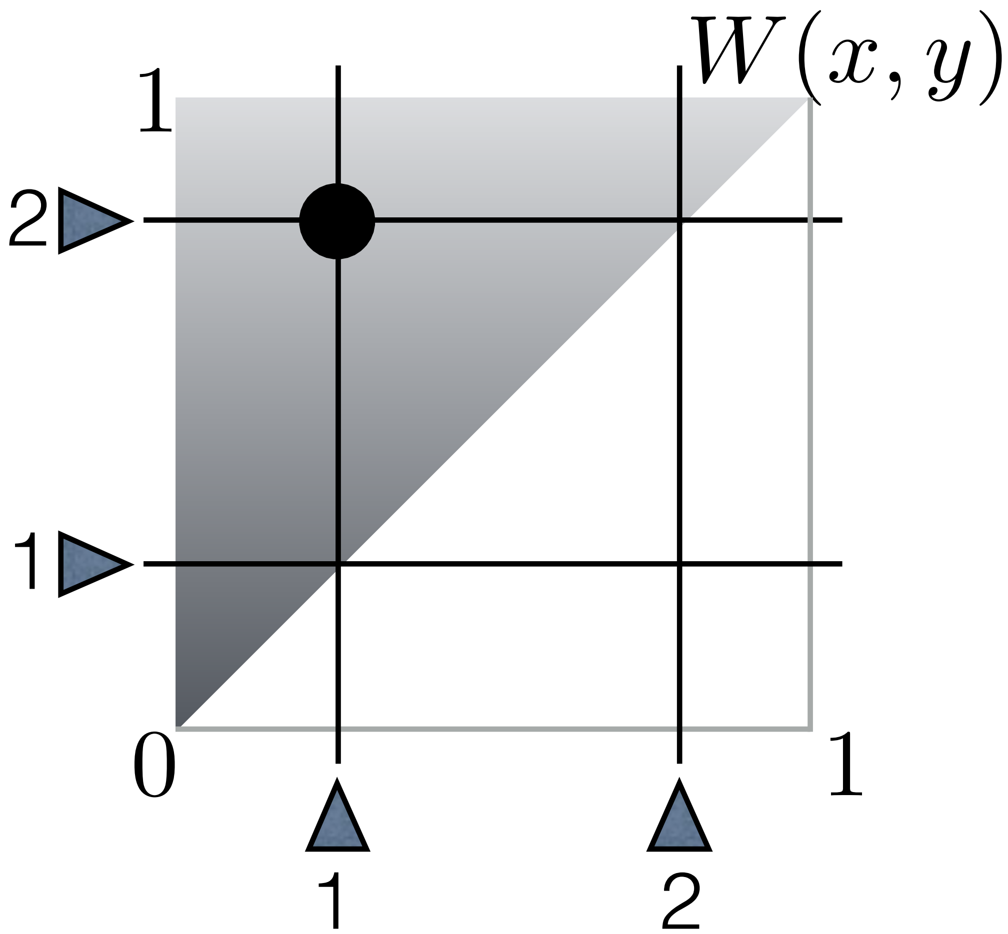

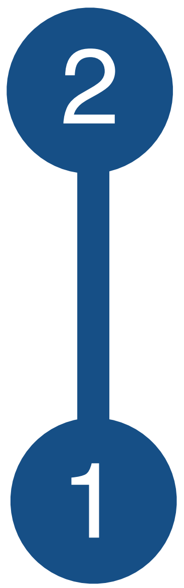

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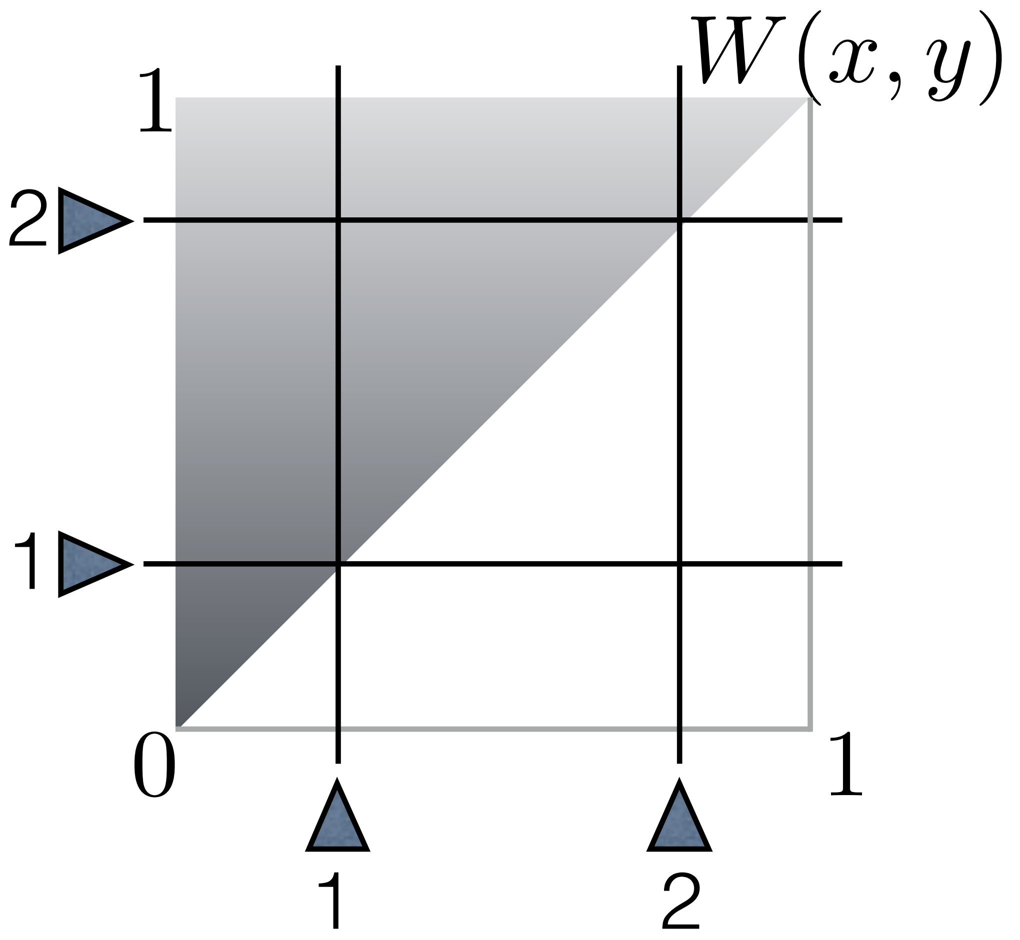

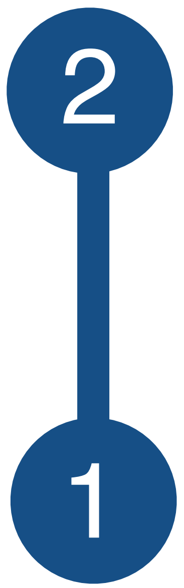

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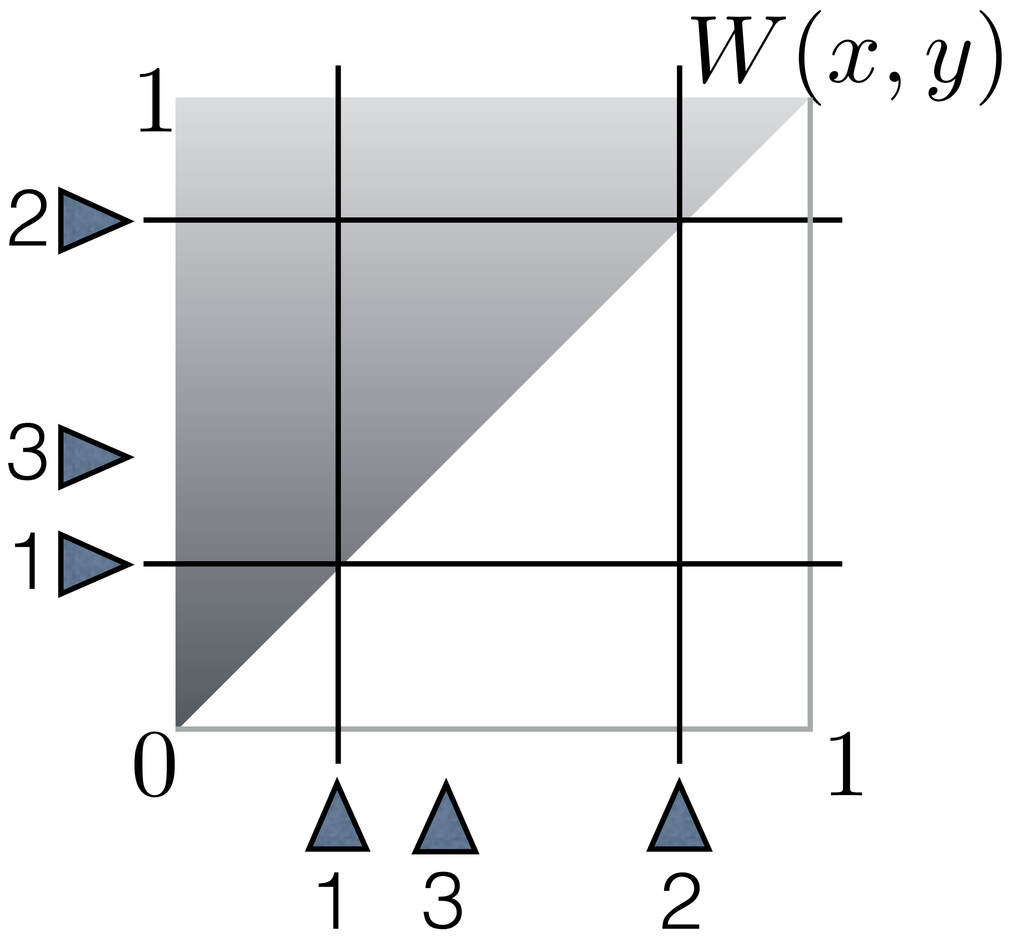

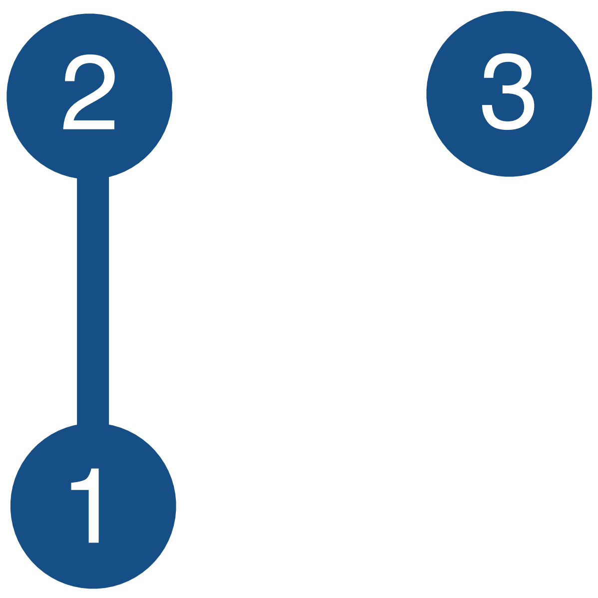

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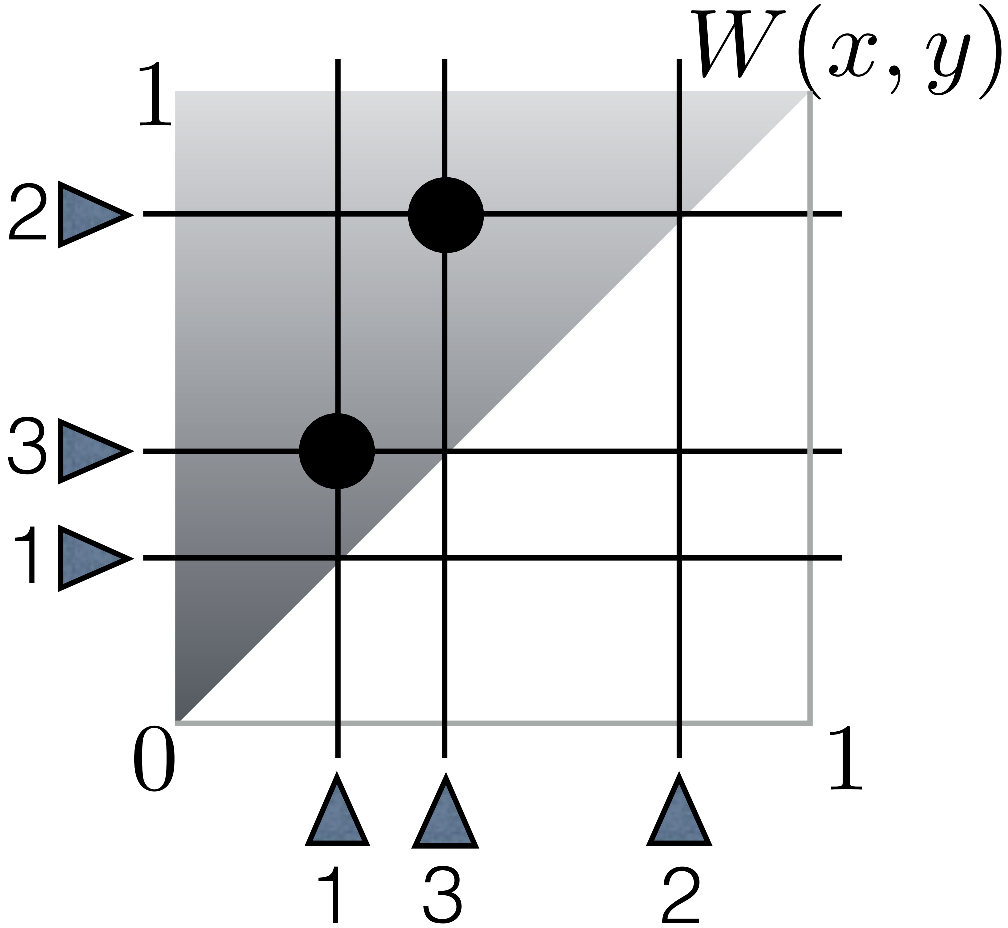

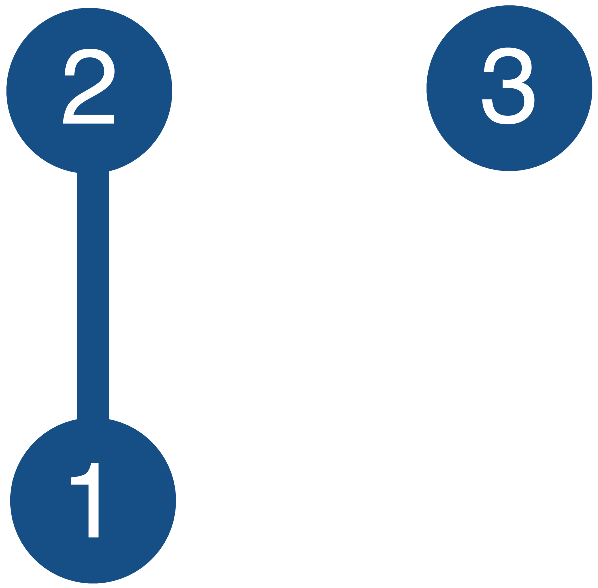

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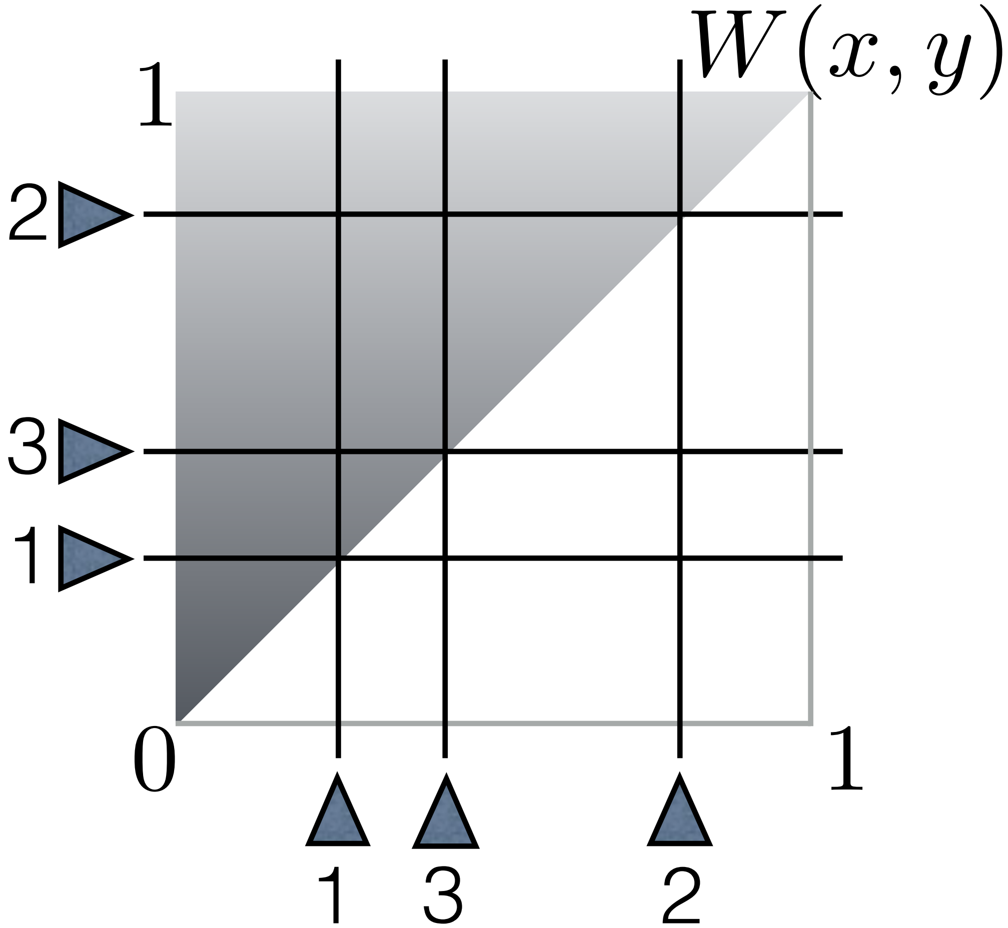

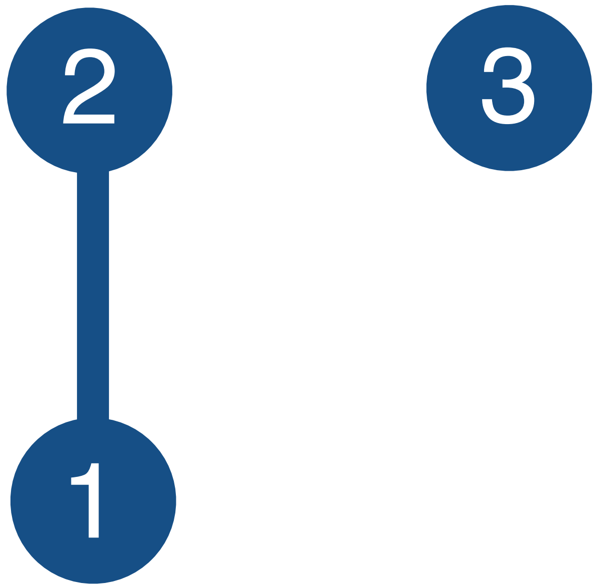

$\mathbb{E}[\#\text{edges}(G_n)]$

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

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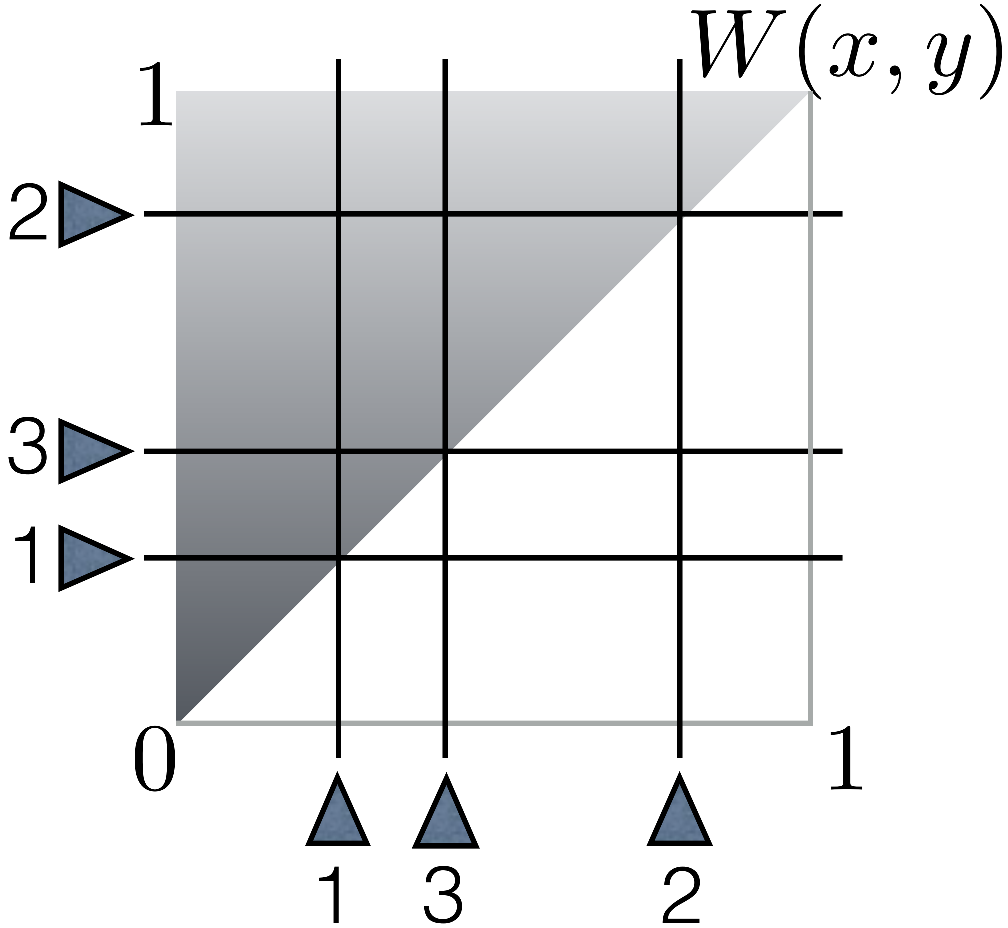
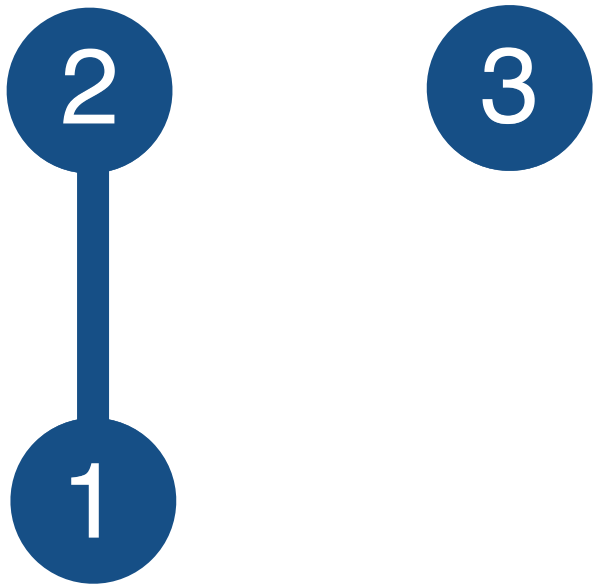
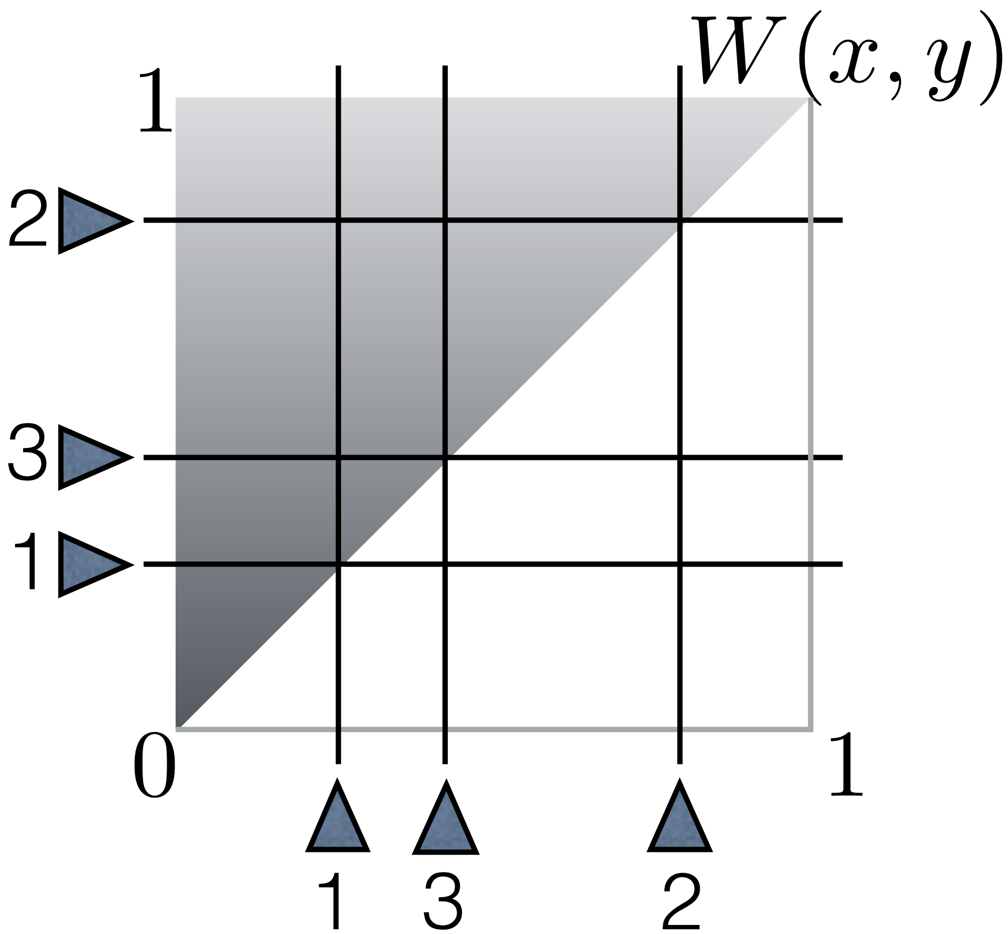


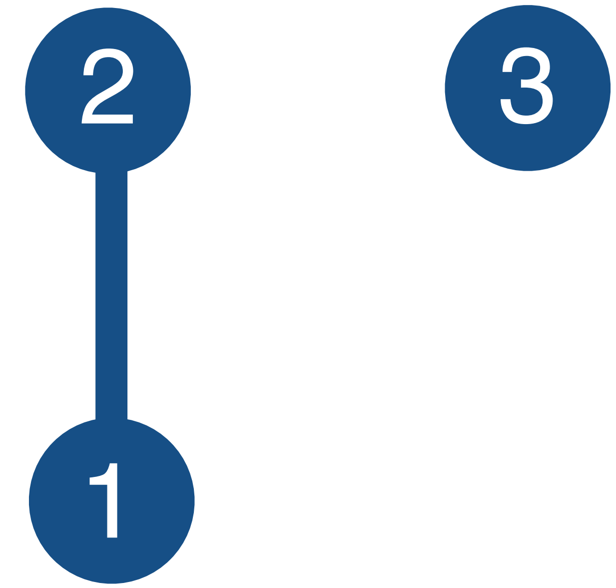


$$\mathbb{E}[\#\text{edges}(G_n)] = \binom{n}{2} \frac{1}{2} \int_0^1 \int_0^1 W(x,y)dx\, dy$$

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

Aldous-Hoover





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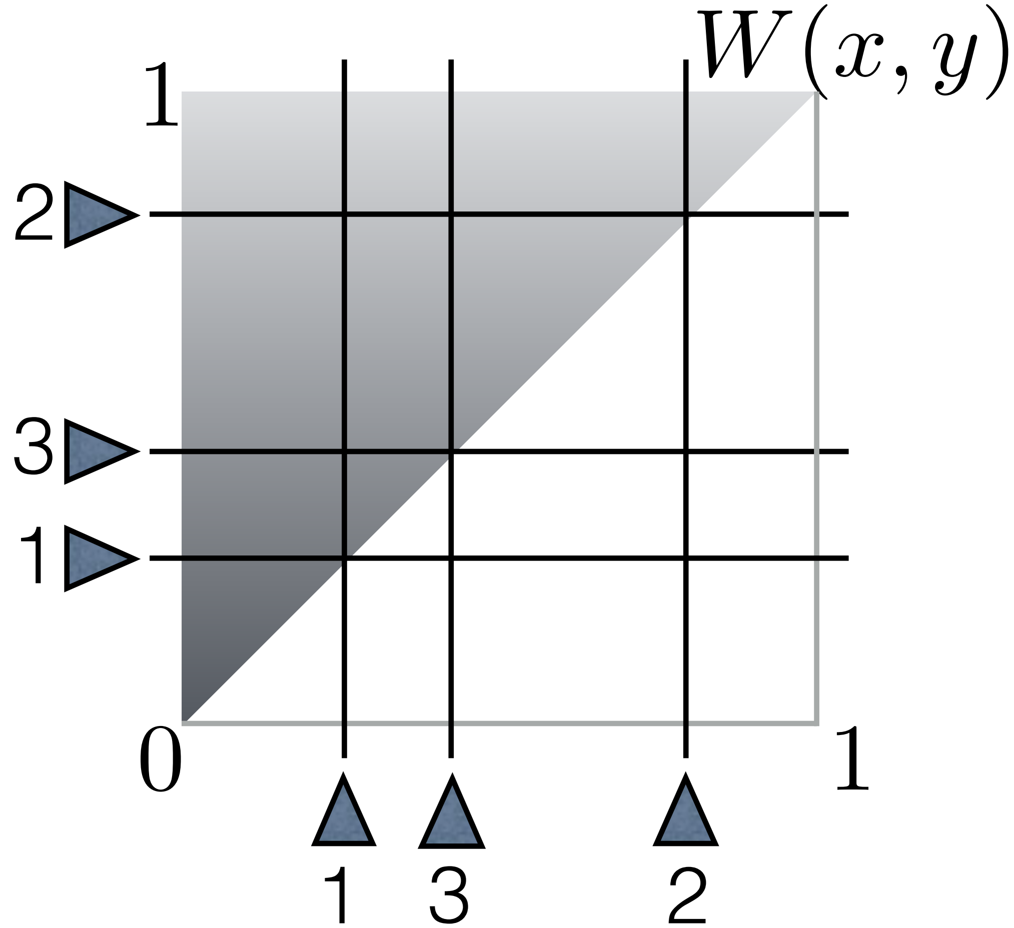

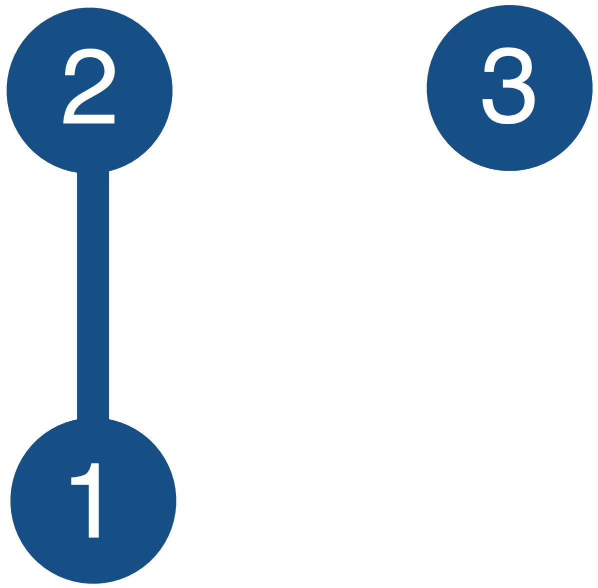

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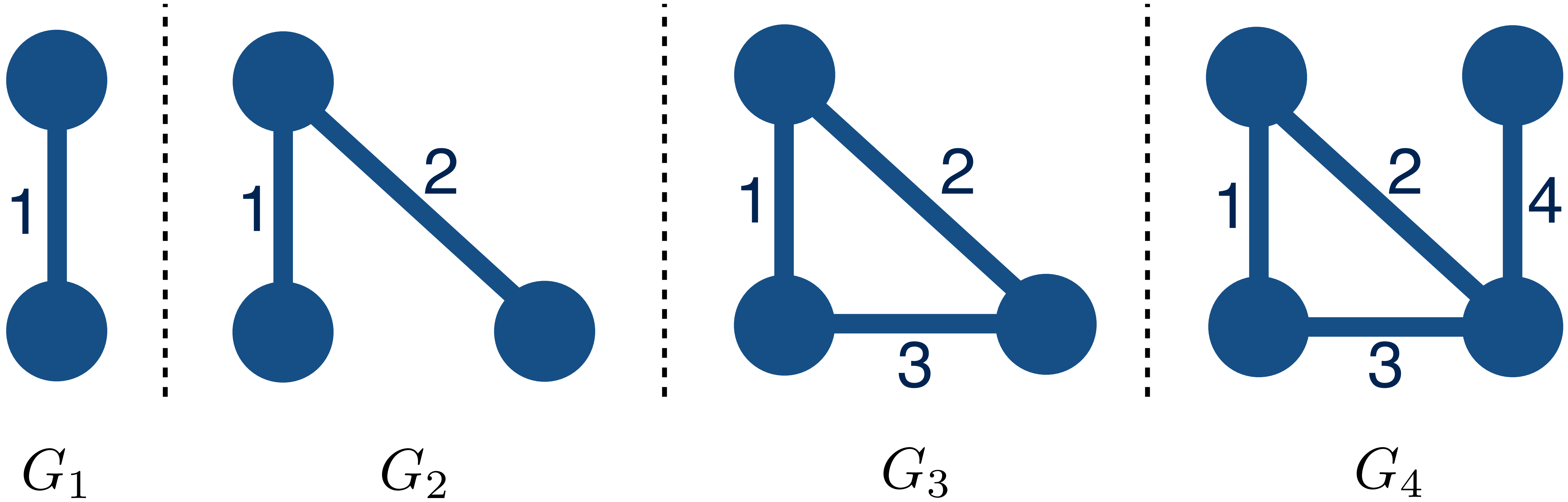
(Might be empty)

[Hoover 1979, Aldous 1981, Orbanz, Roy 2015]

Sequence of graphs



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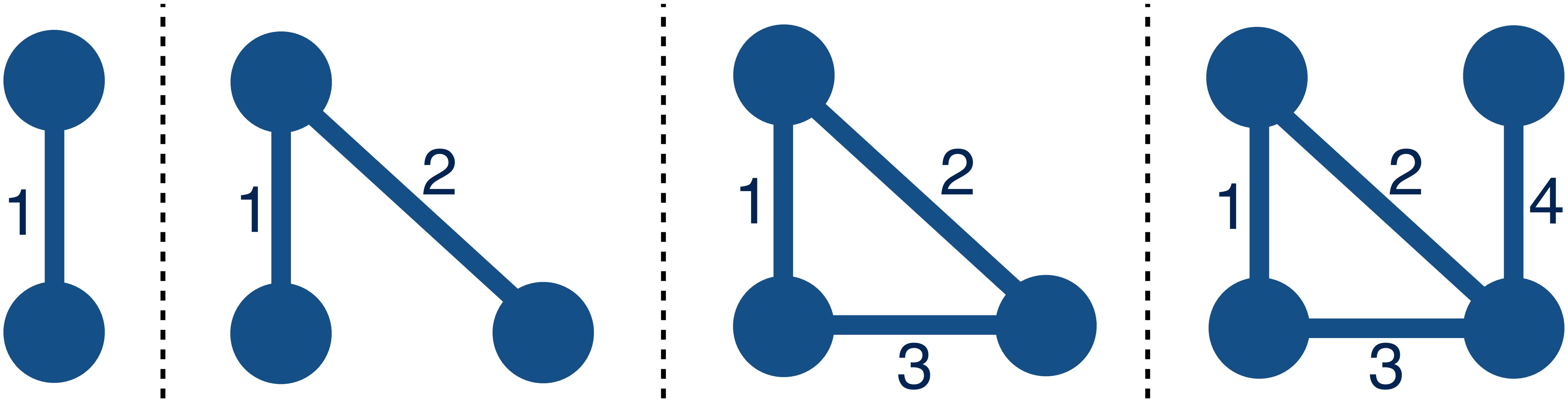
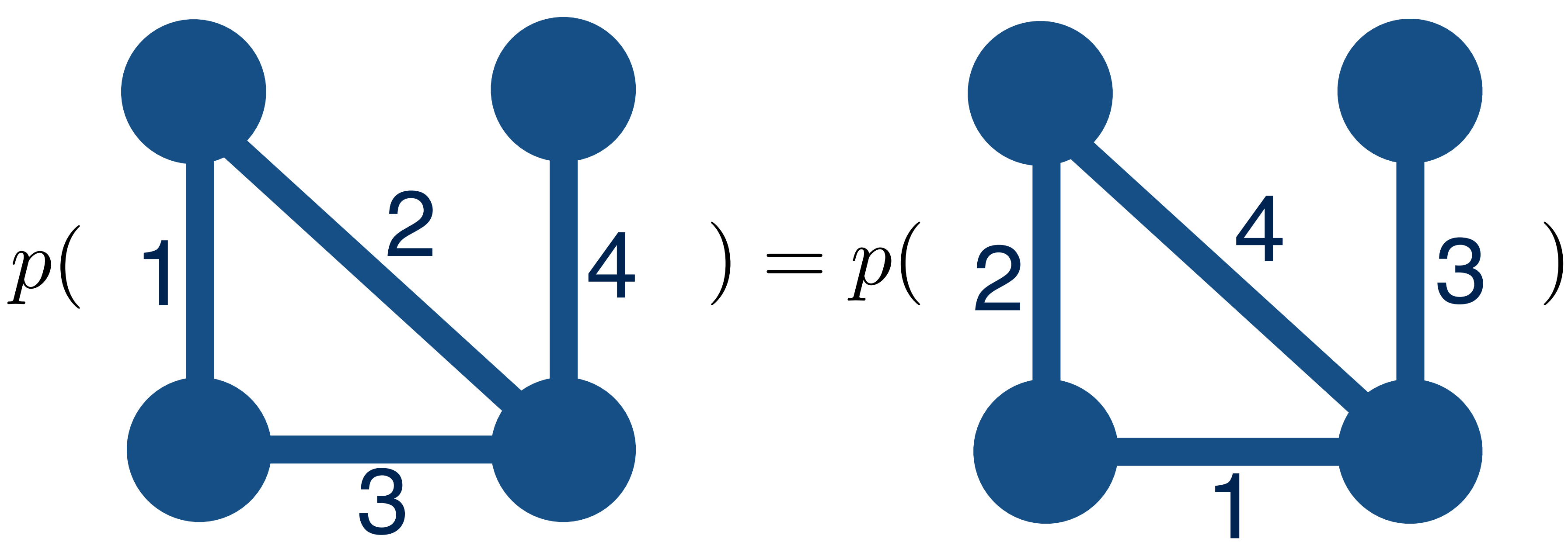
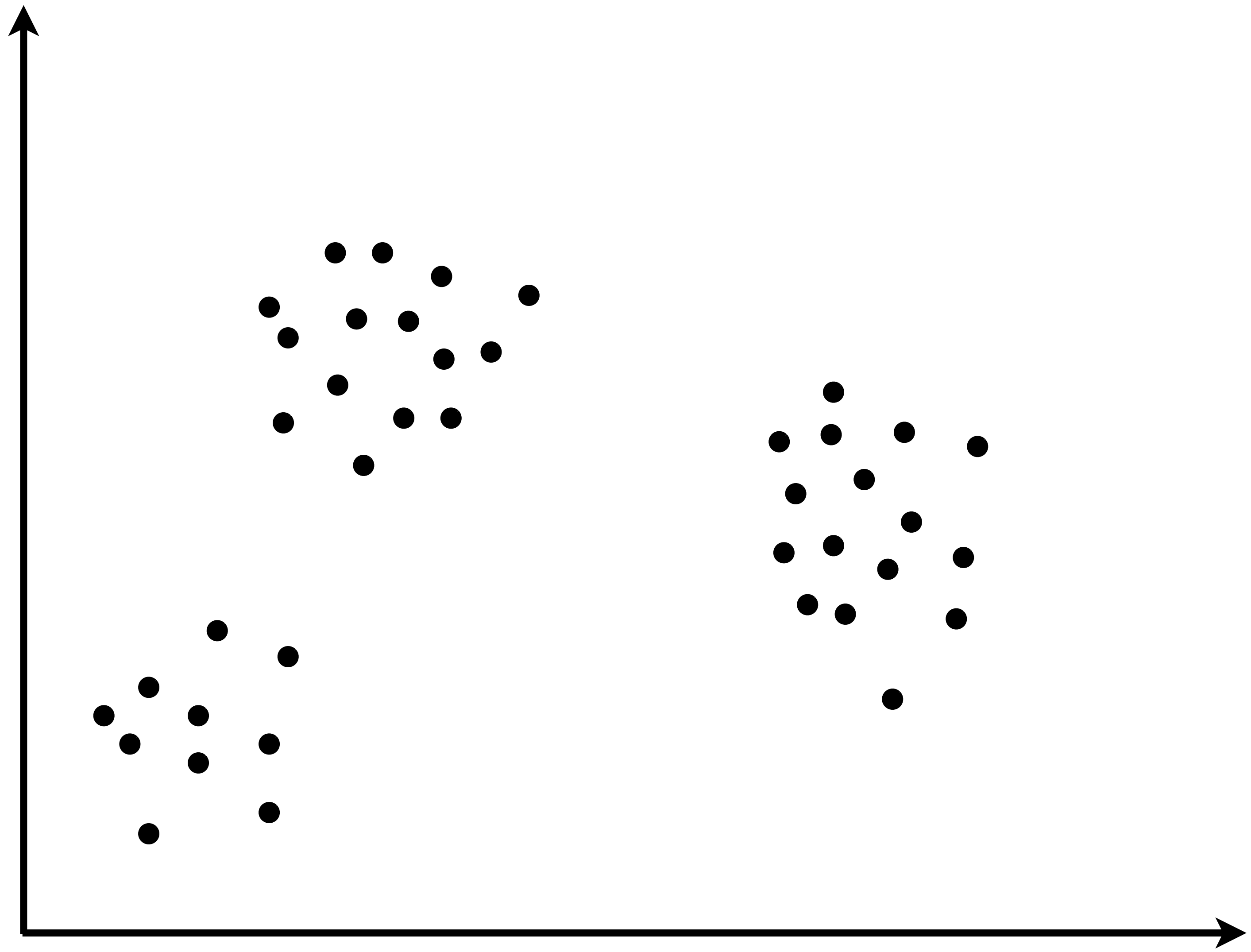


G_1 G_2 G_3 G_4



Clustering/partition



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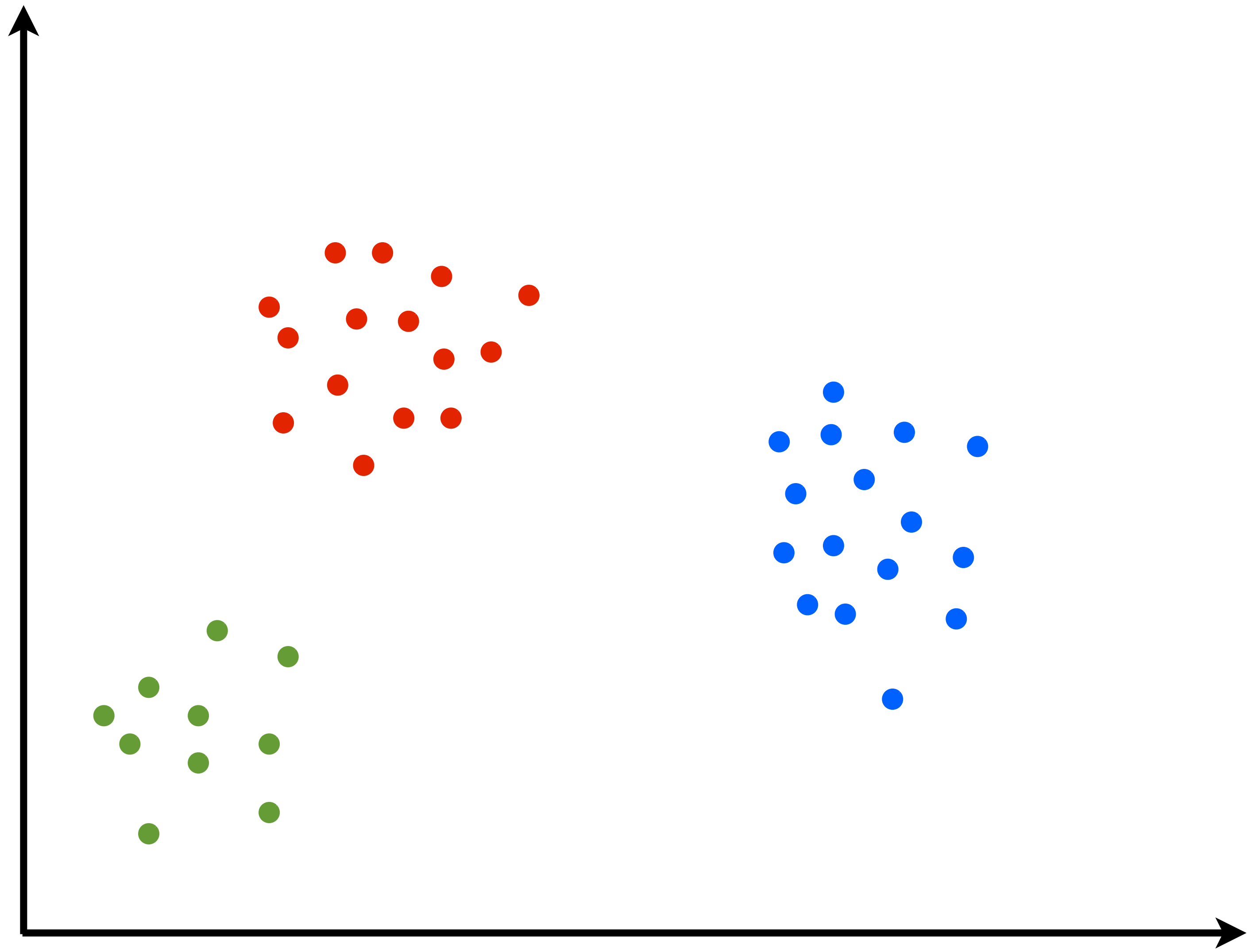
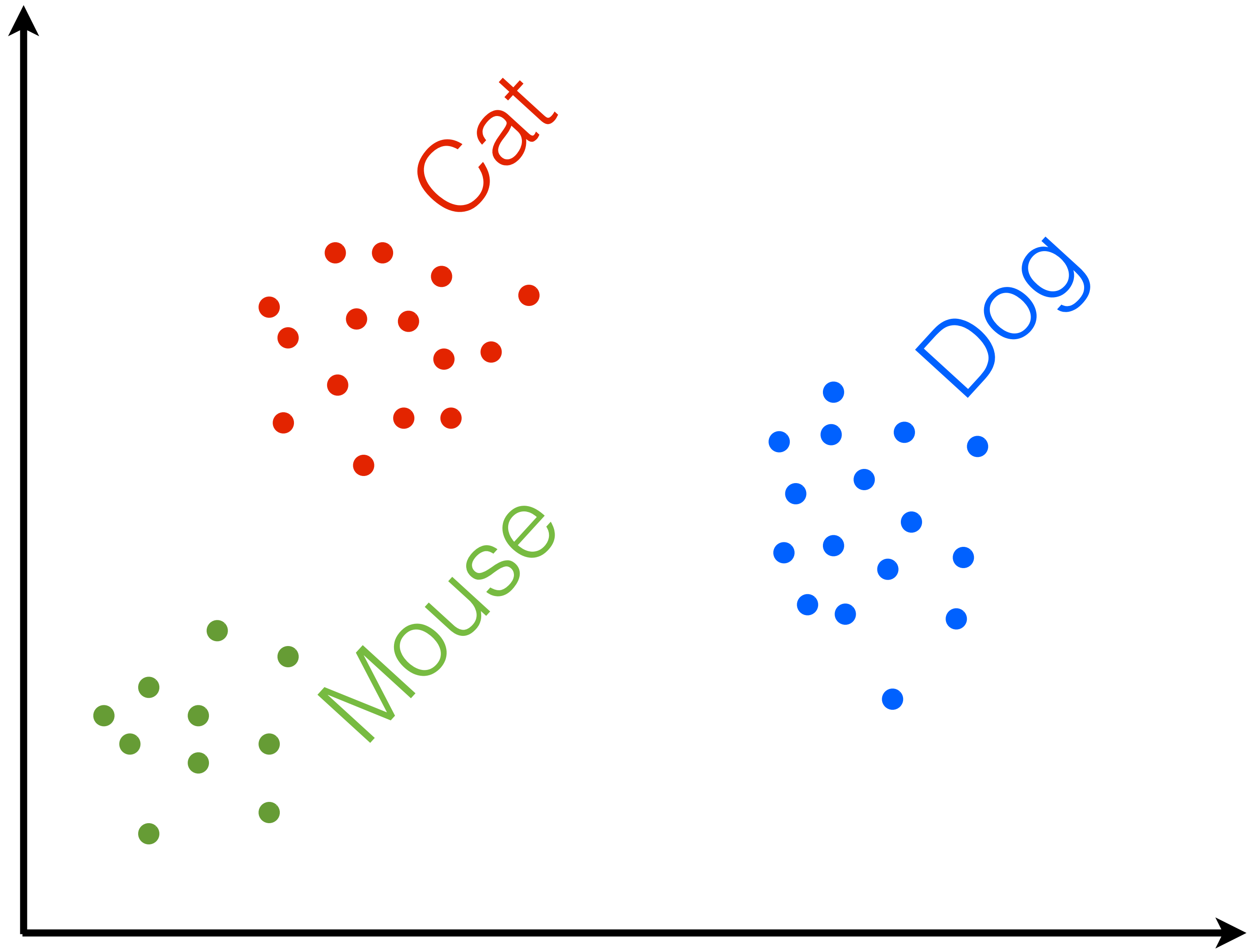


“Clusters”,
“classes”,
“blocks of a partition”

Clustering/partition



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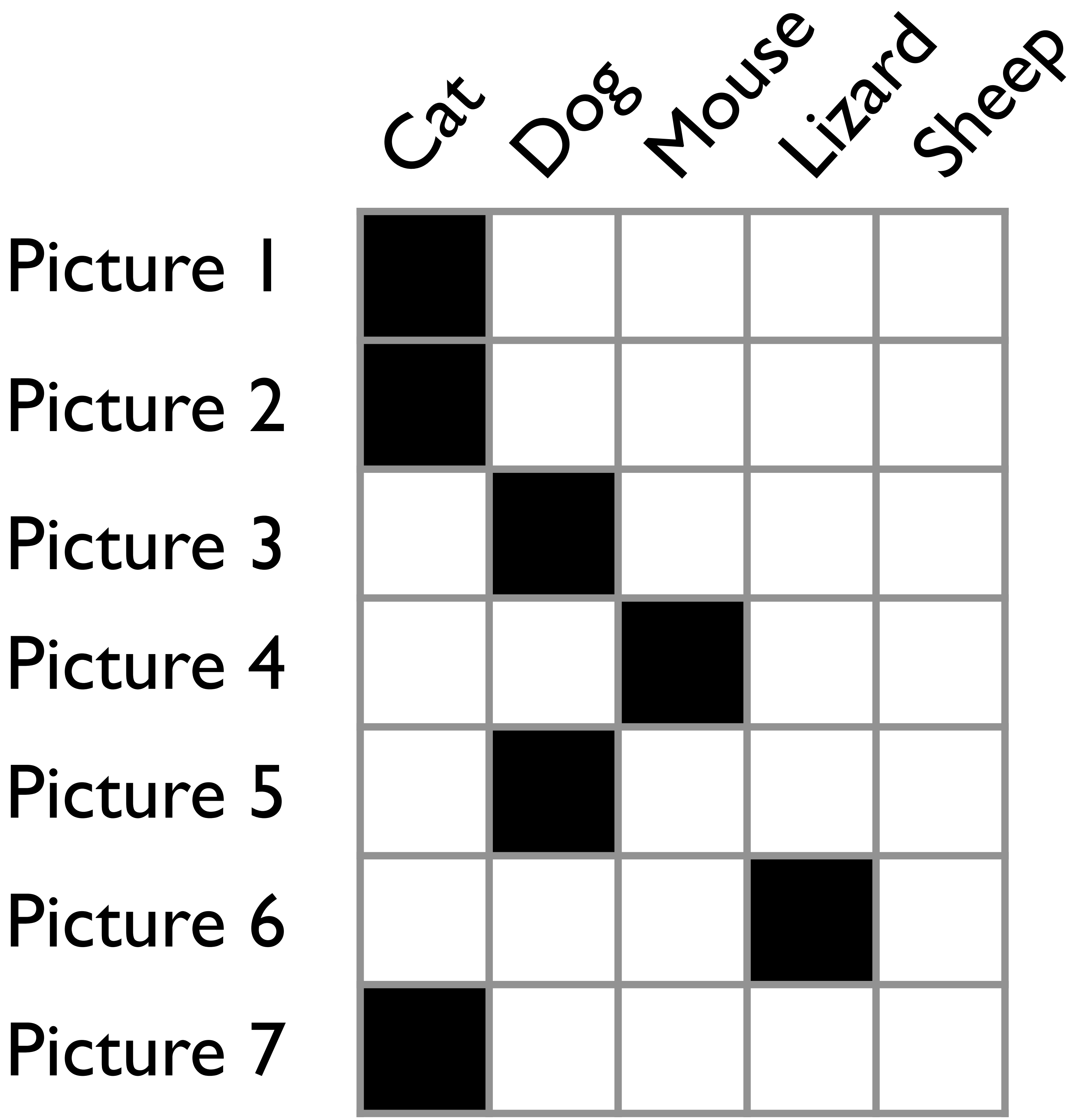
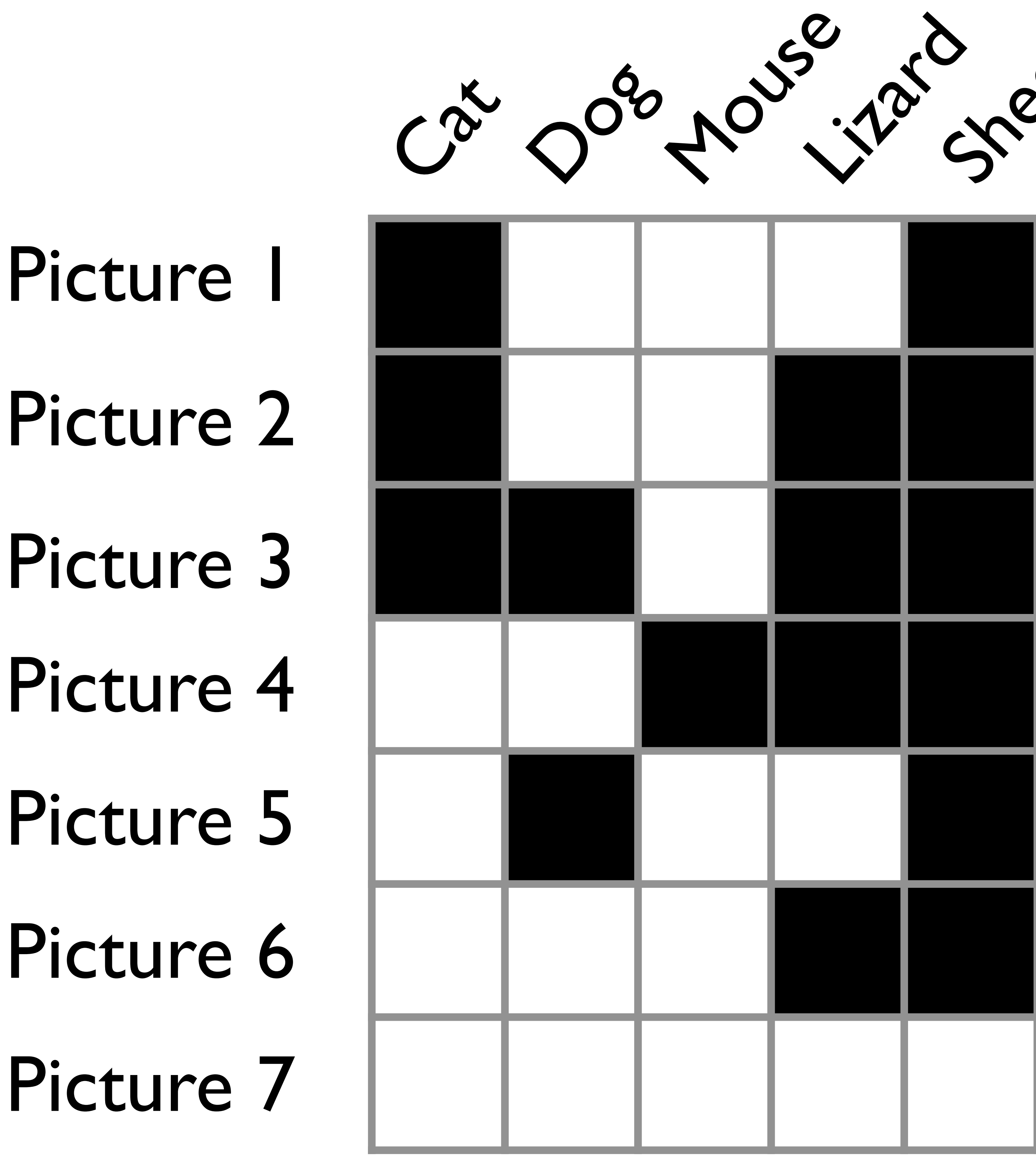
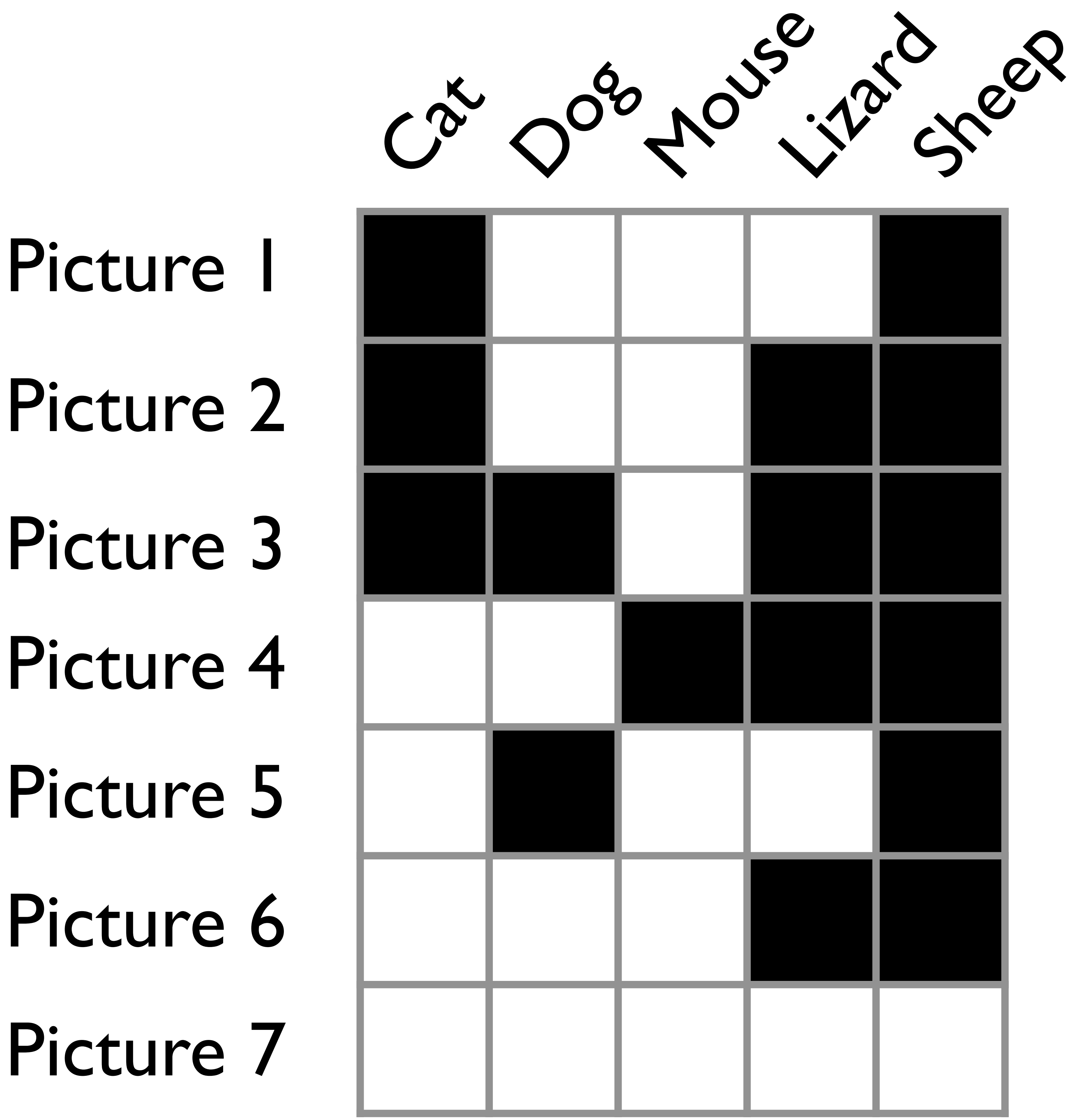


Feature allocation



“Features”,
“topics”

Feature allocation



“Features”,
“topics”

- Exchangeable
- Finite # of features per data point

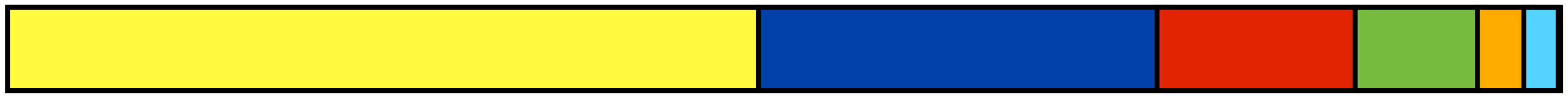
Paintboxes

Exchangeable partition: Kingman paintbox

[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox



[Kingman 1978]

Paintboxes

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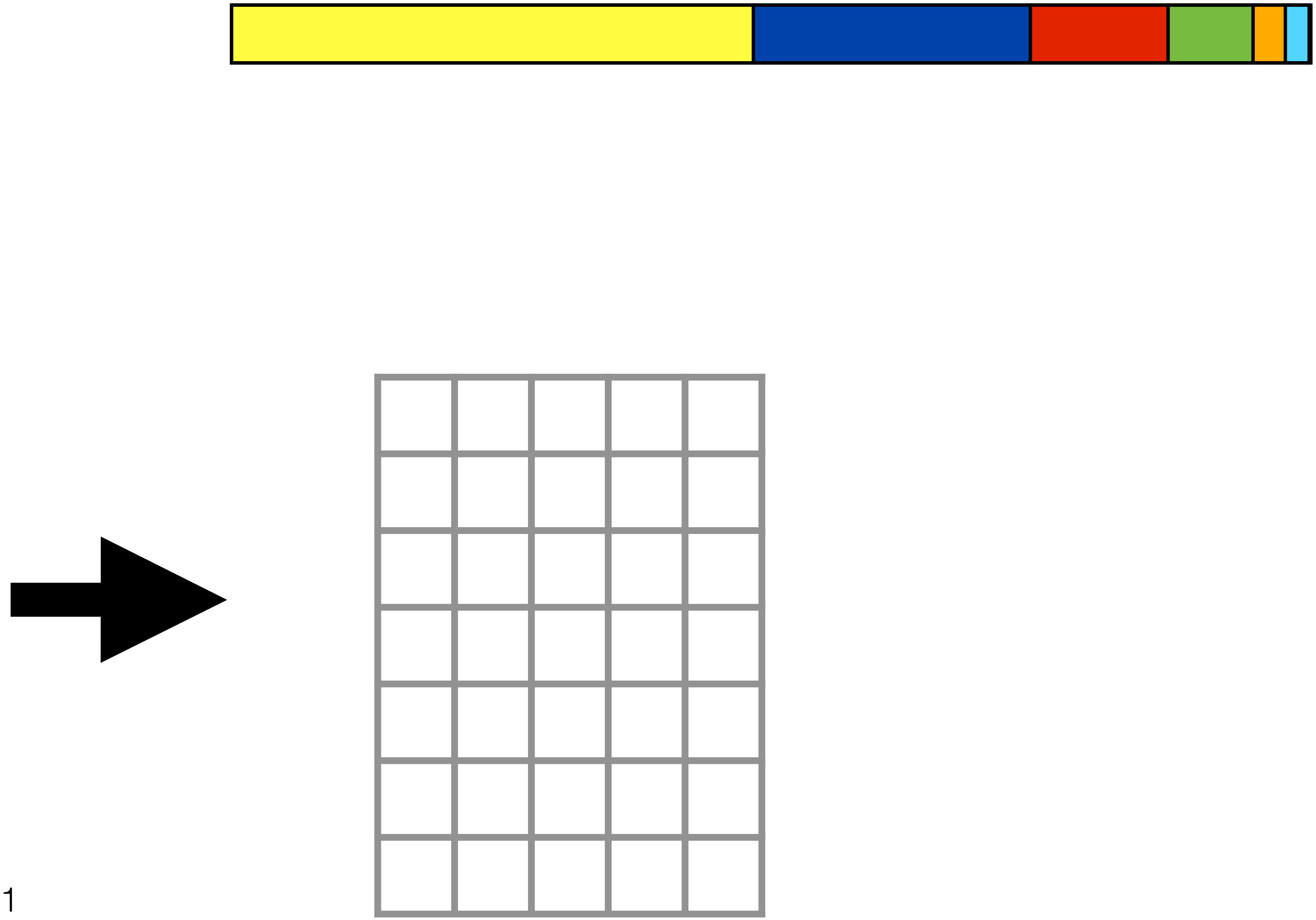

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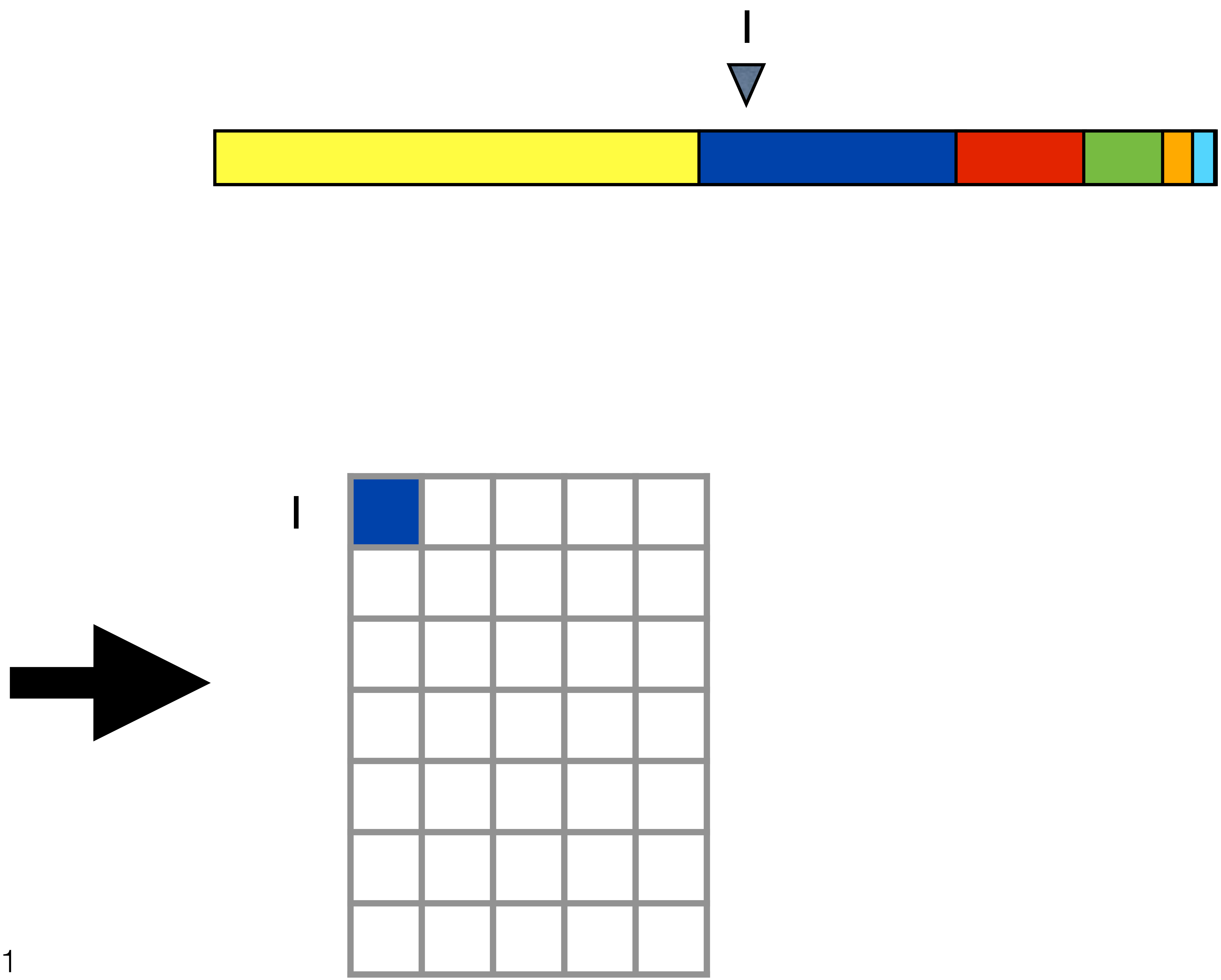

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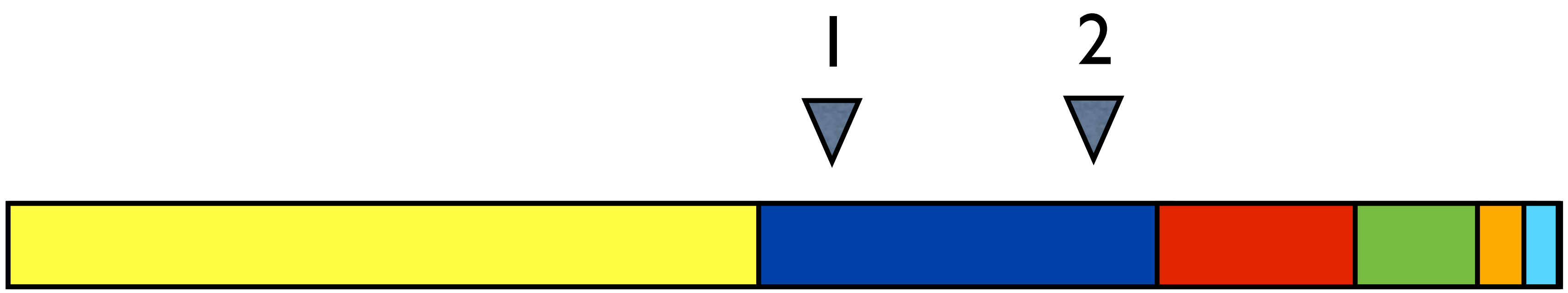

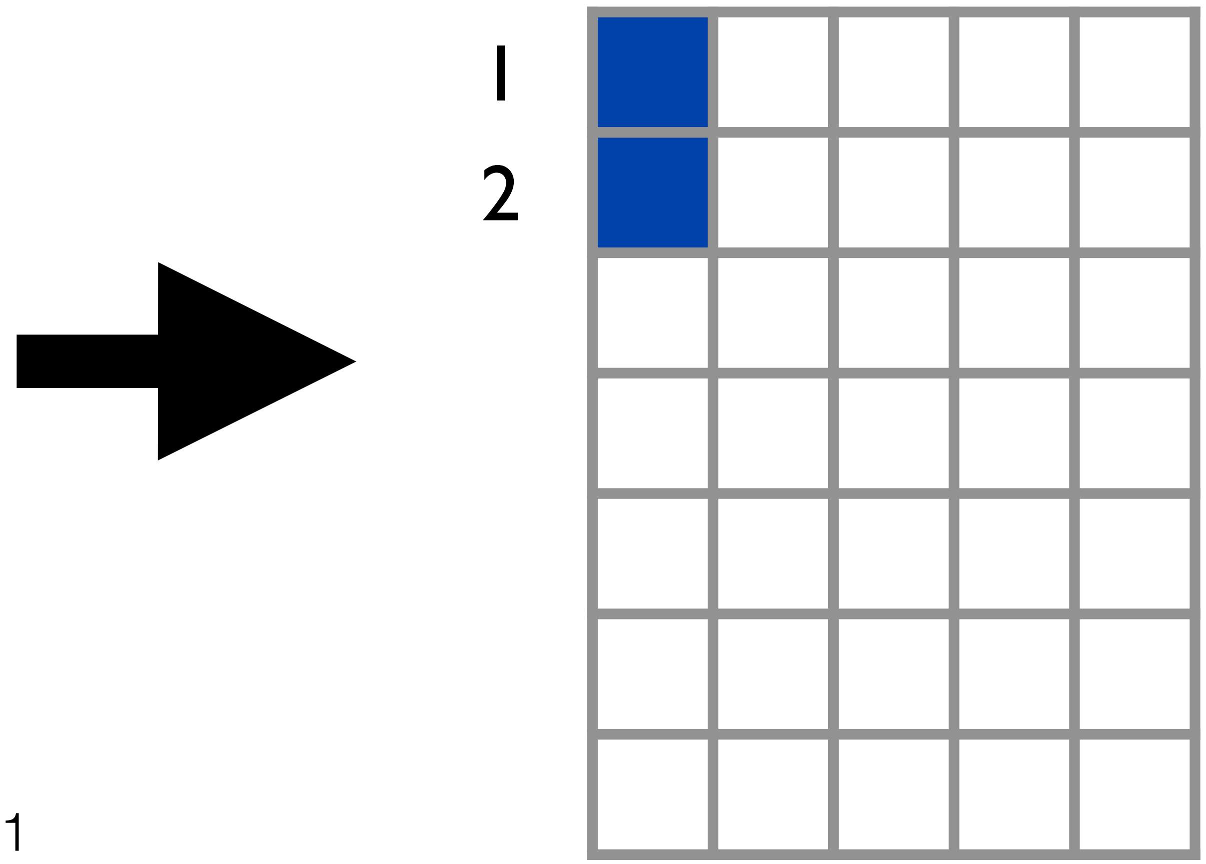

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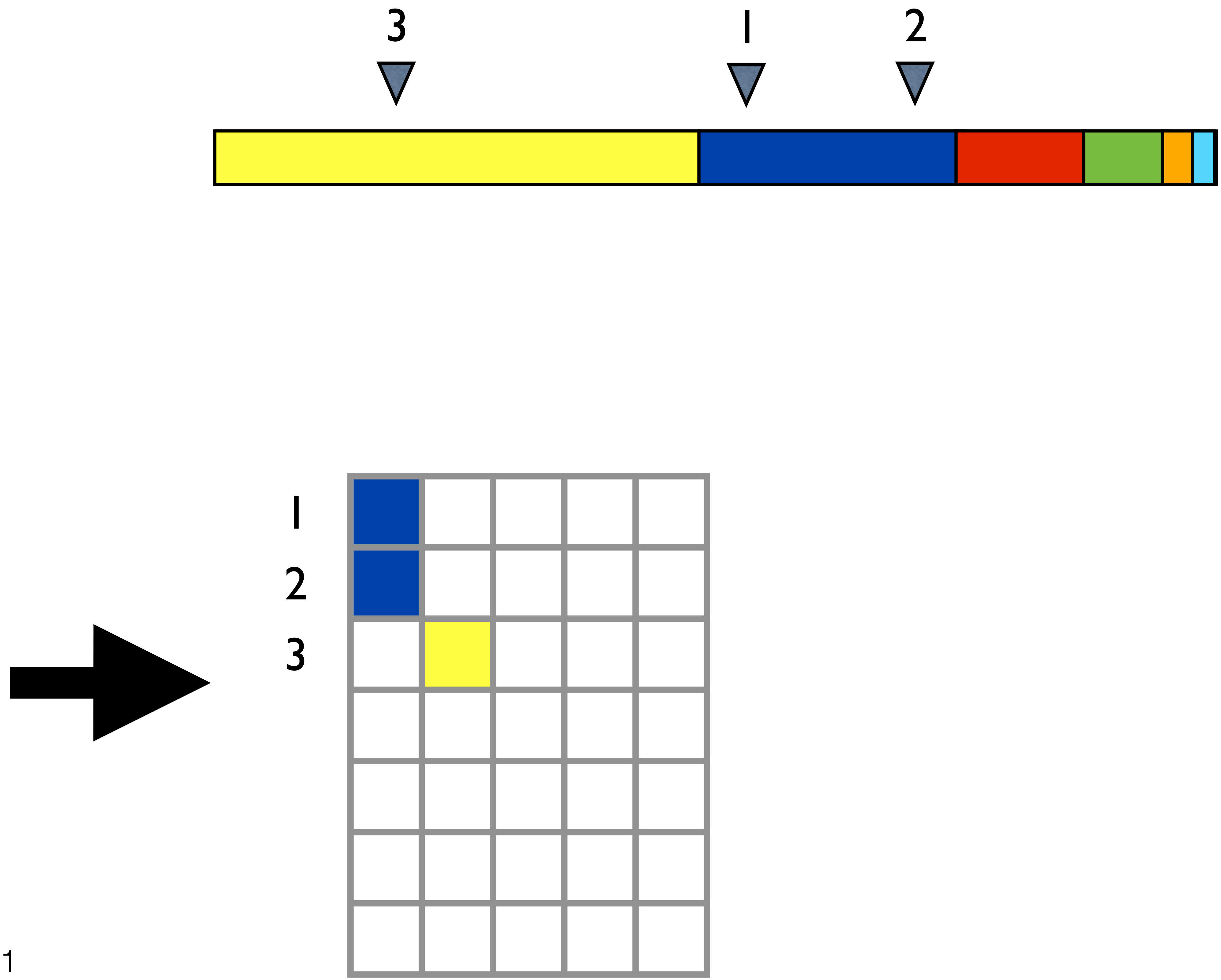

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Paintboxes

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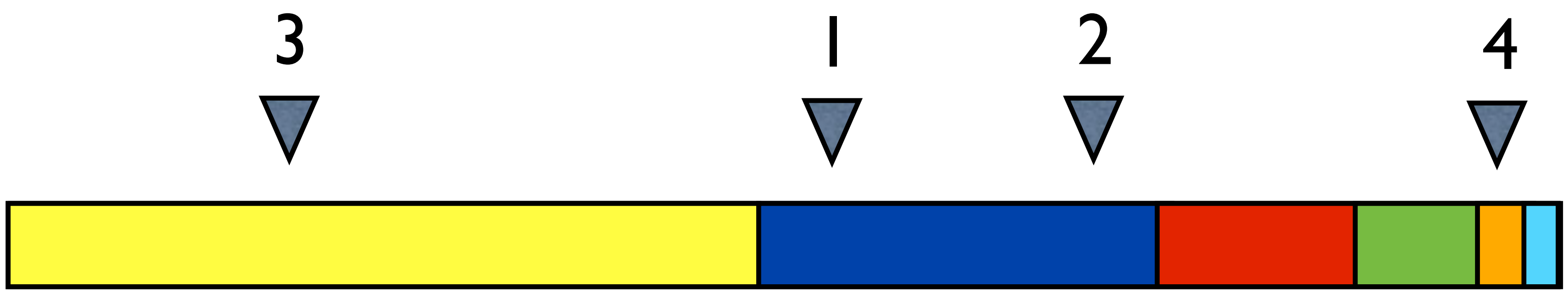

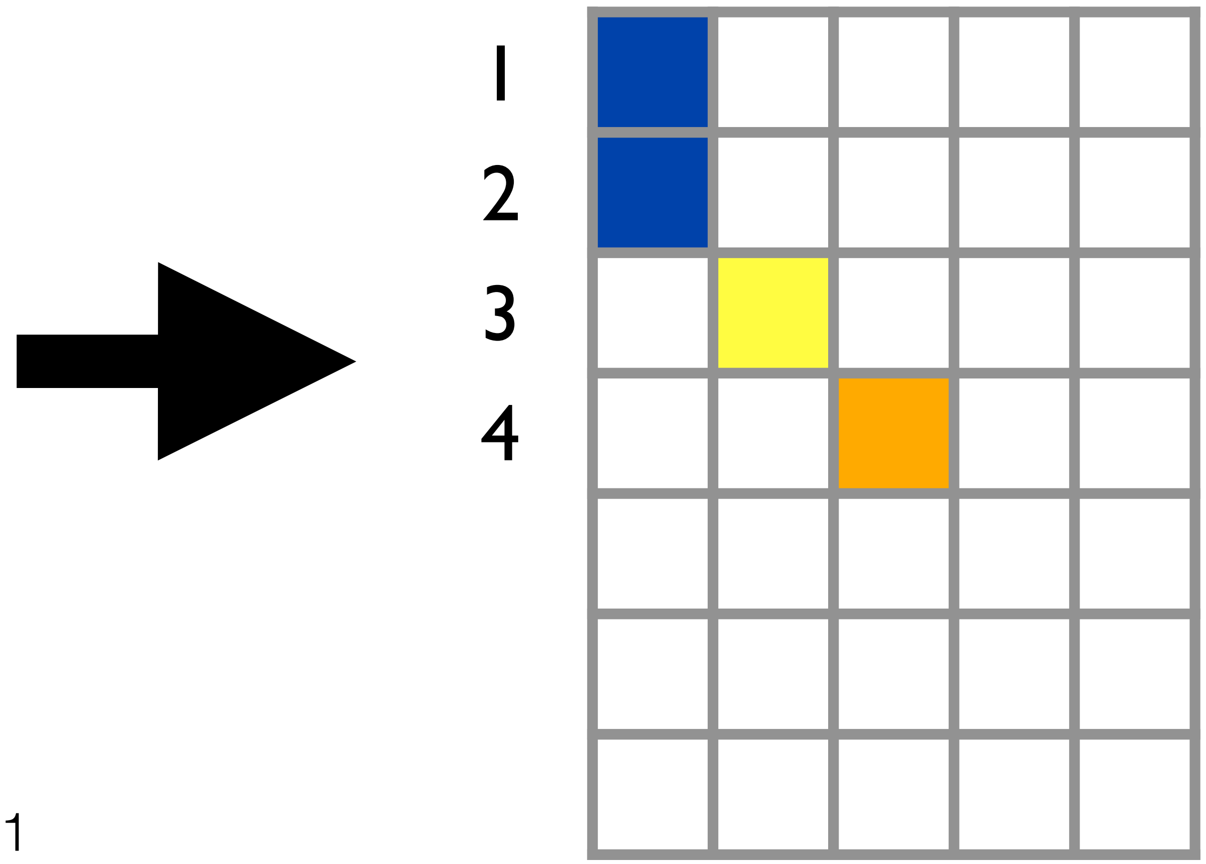

[Kingman 1978]

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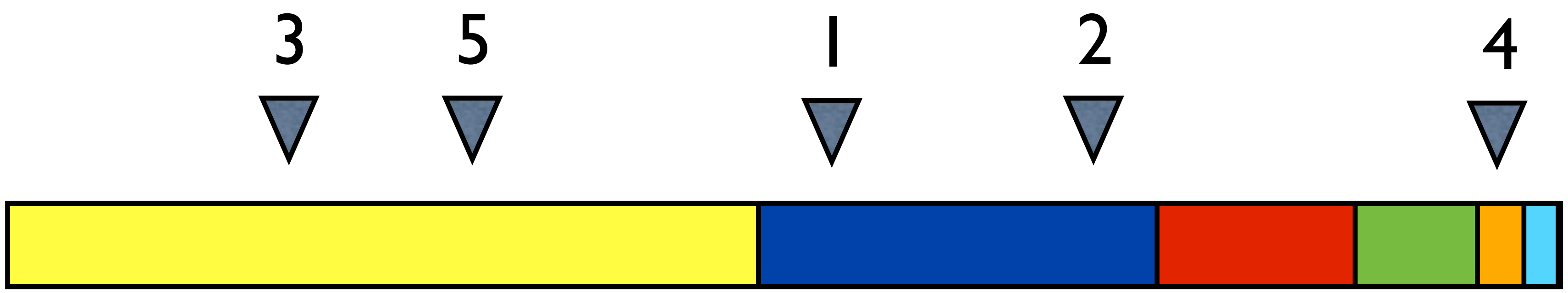

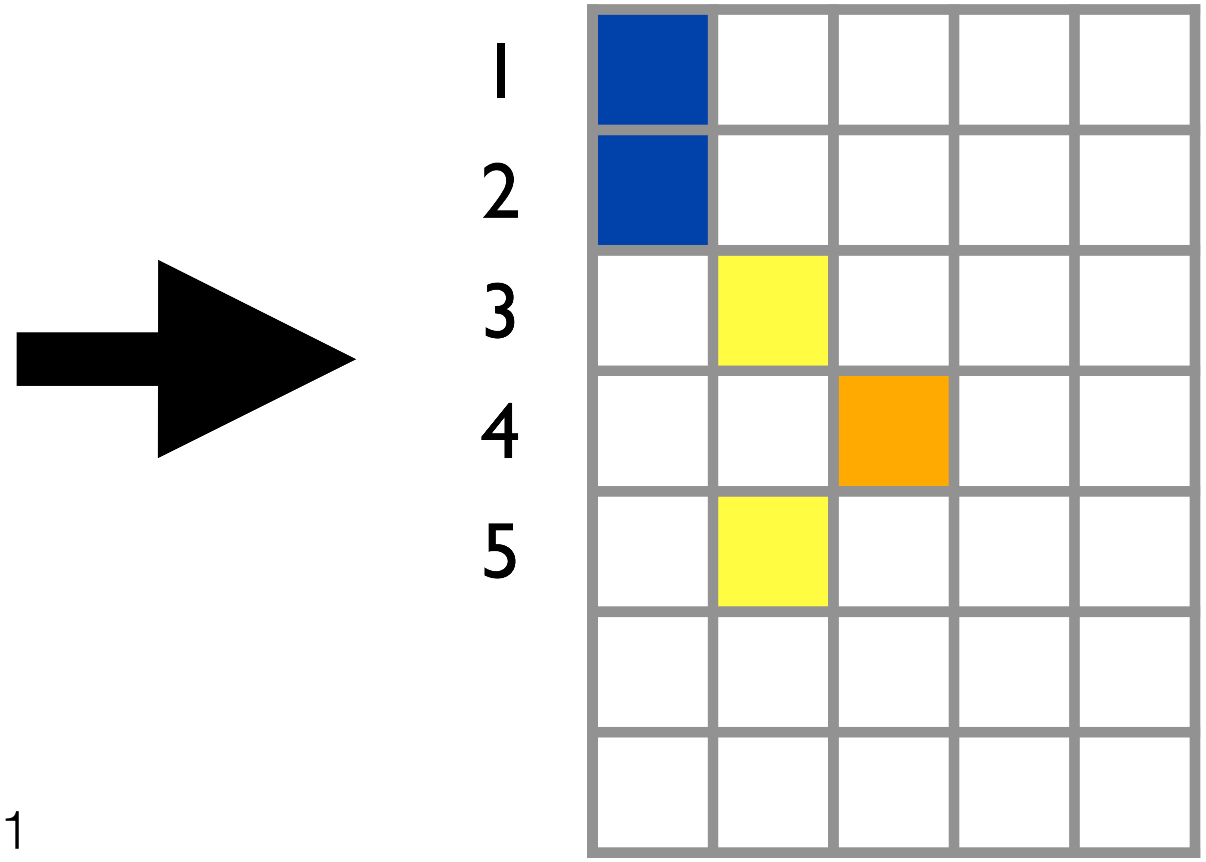

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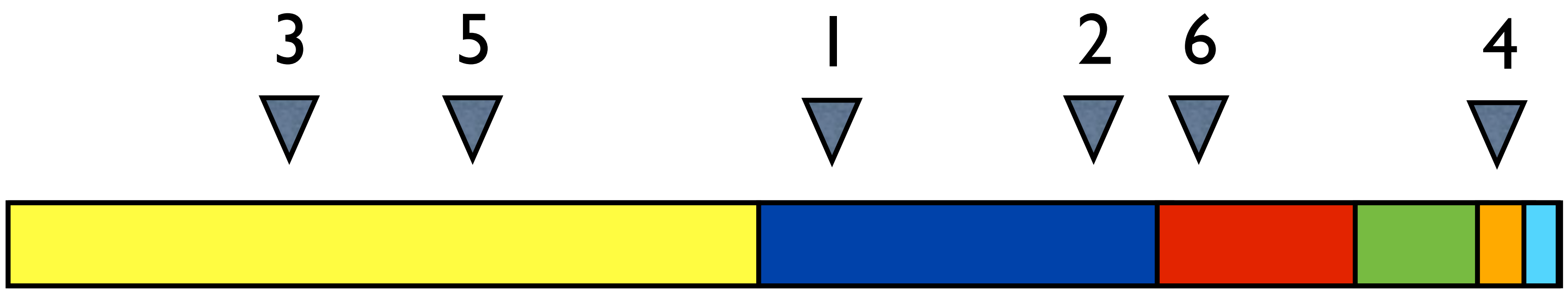

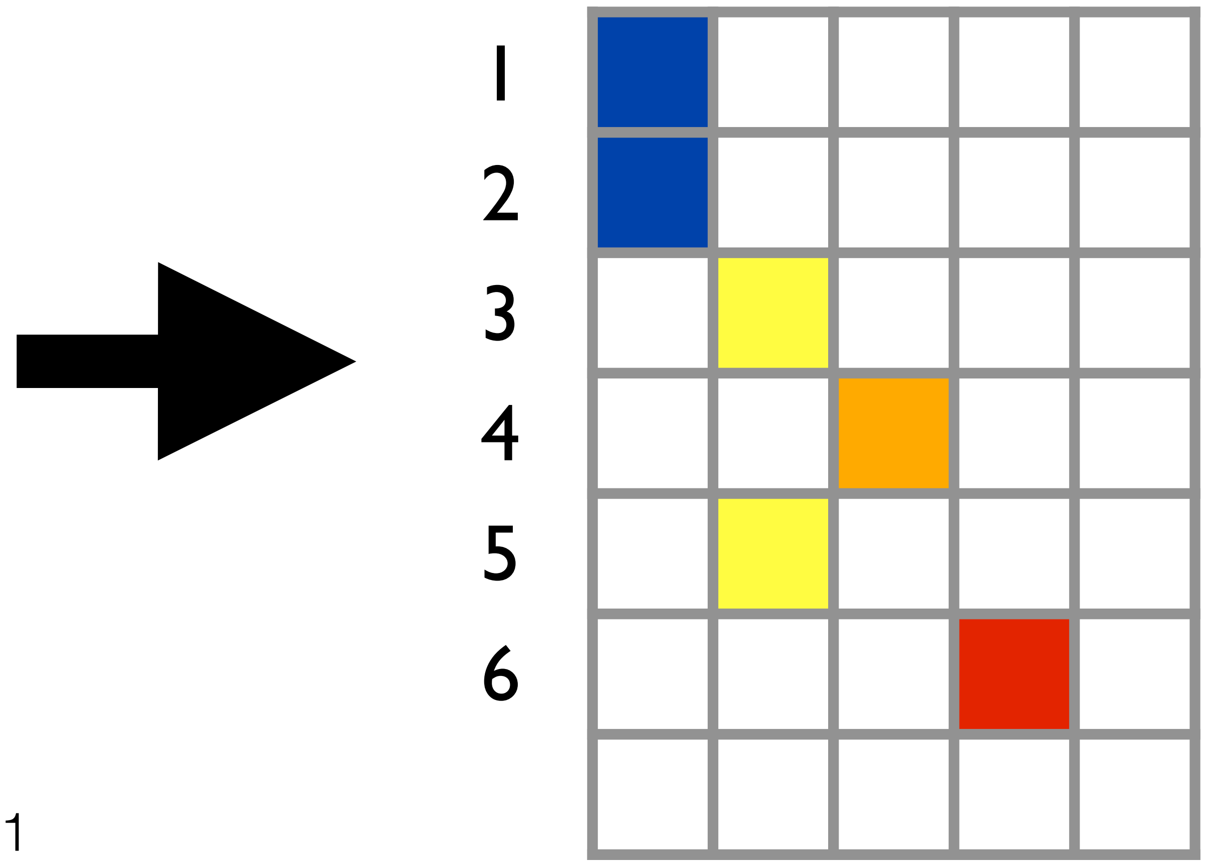

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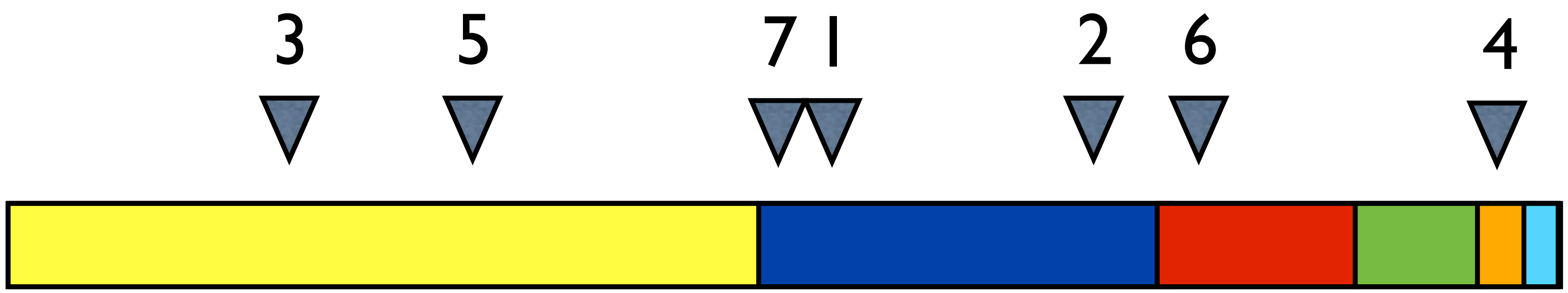

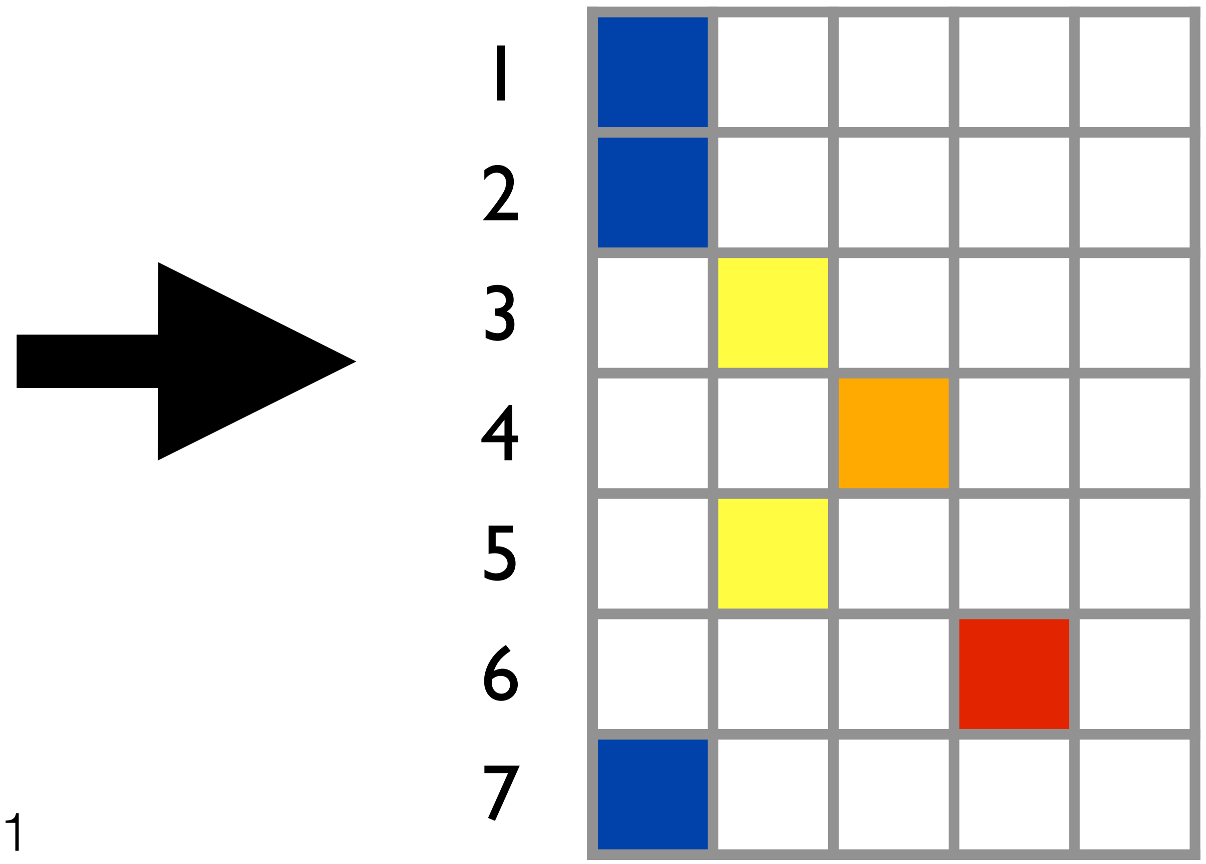

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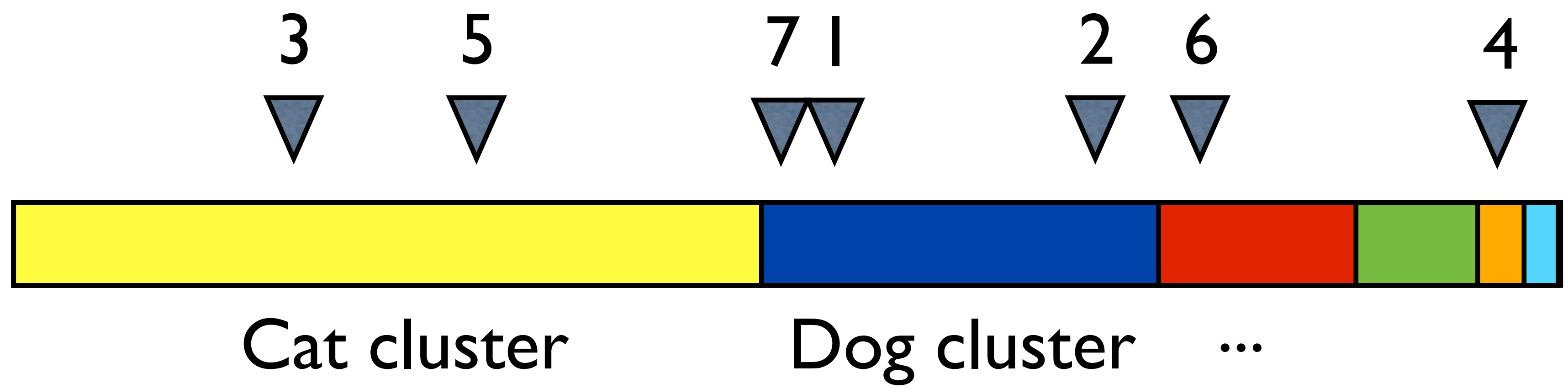

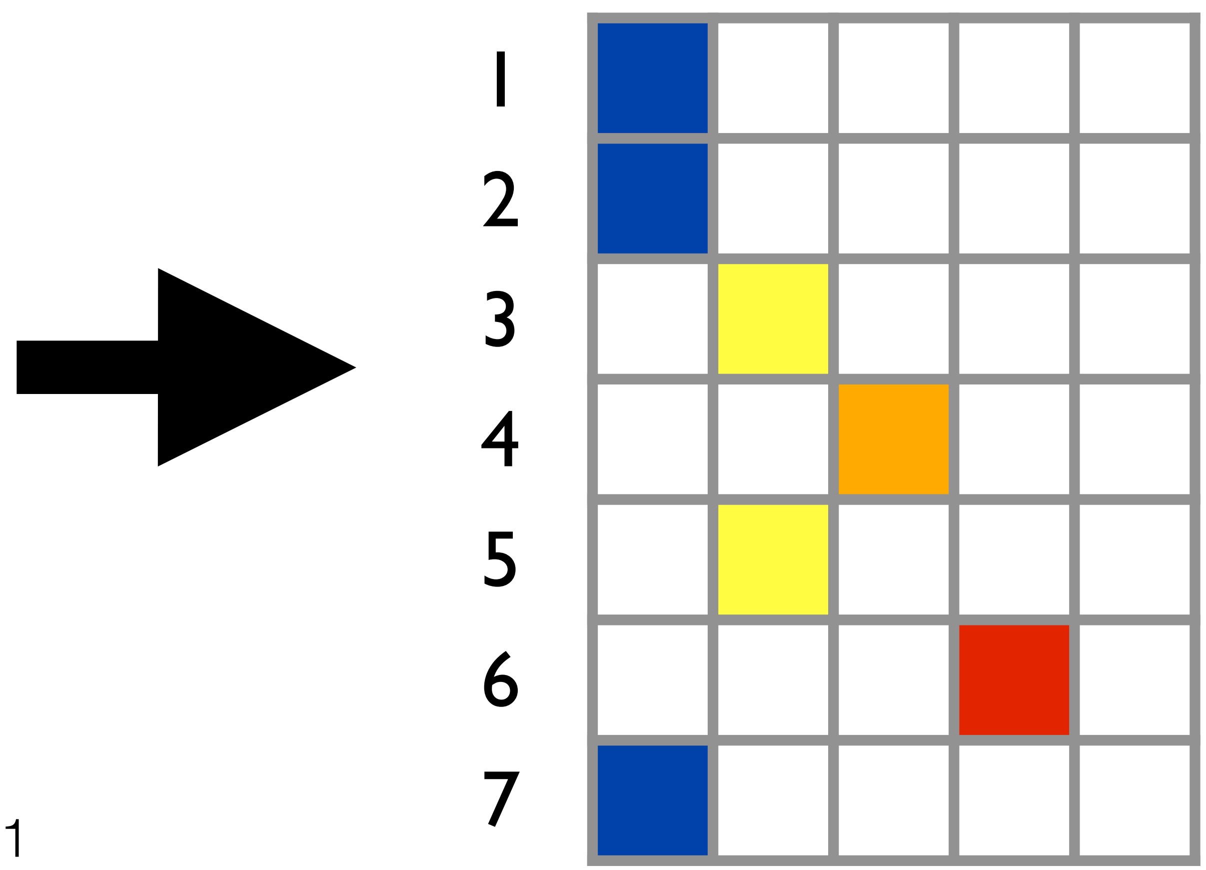

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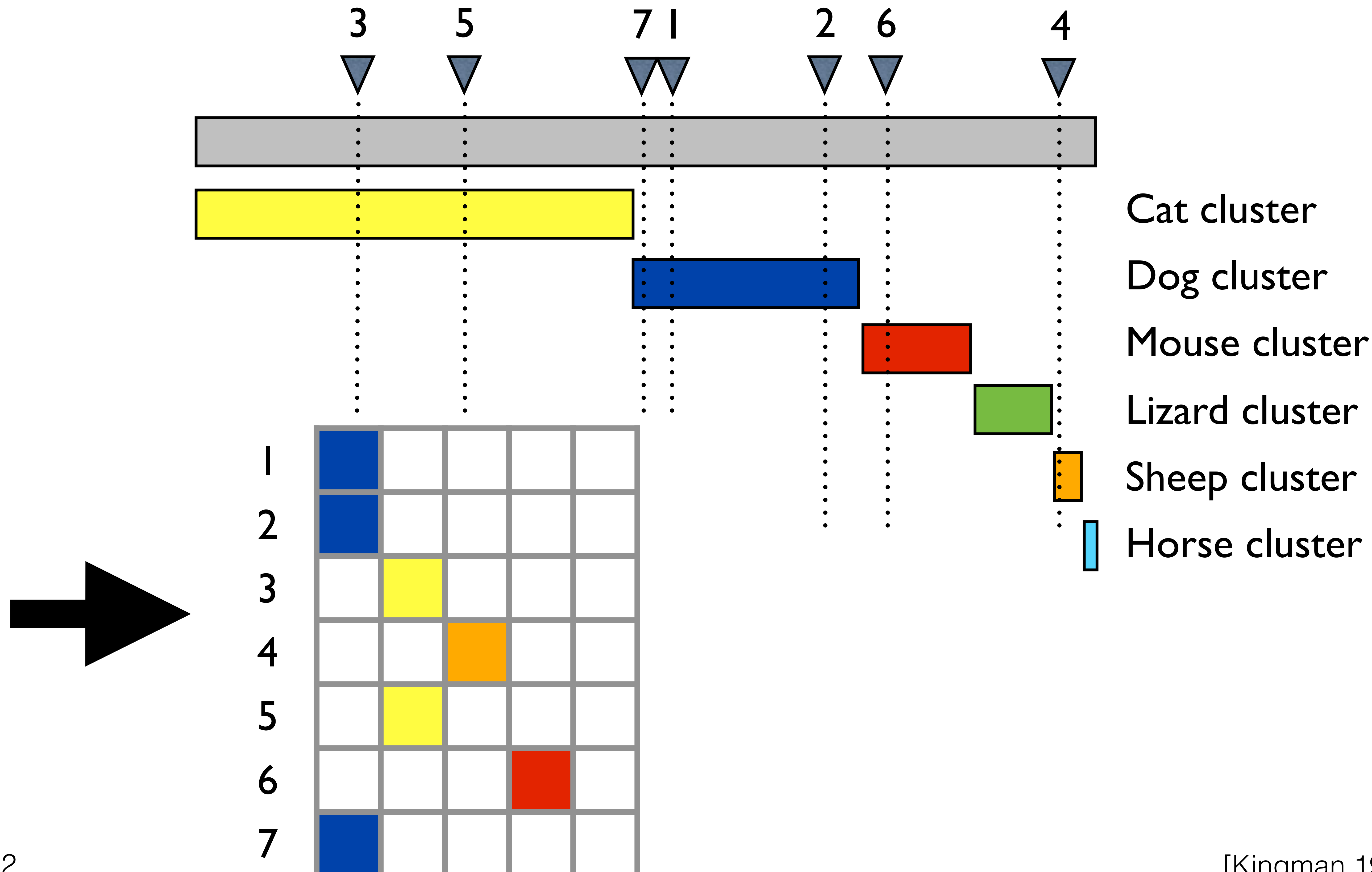

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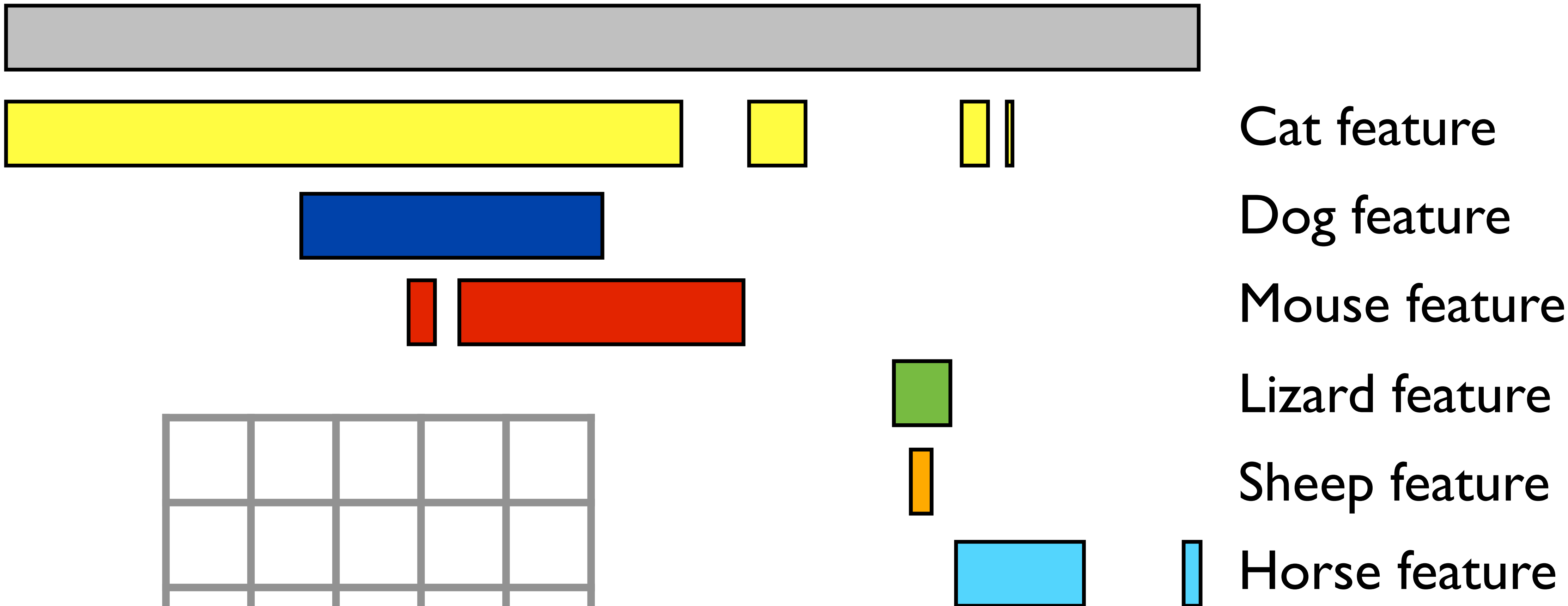

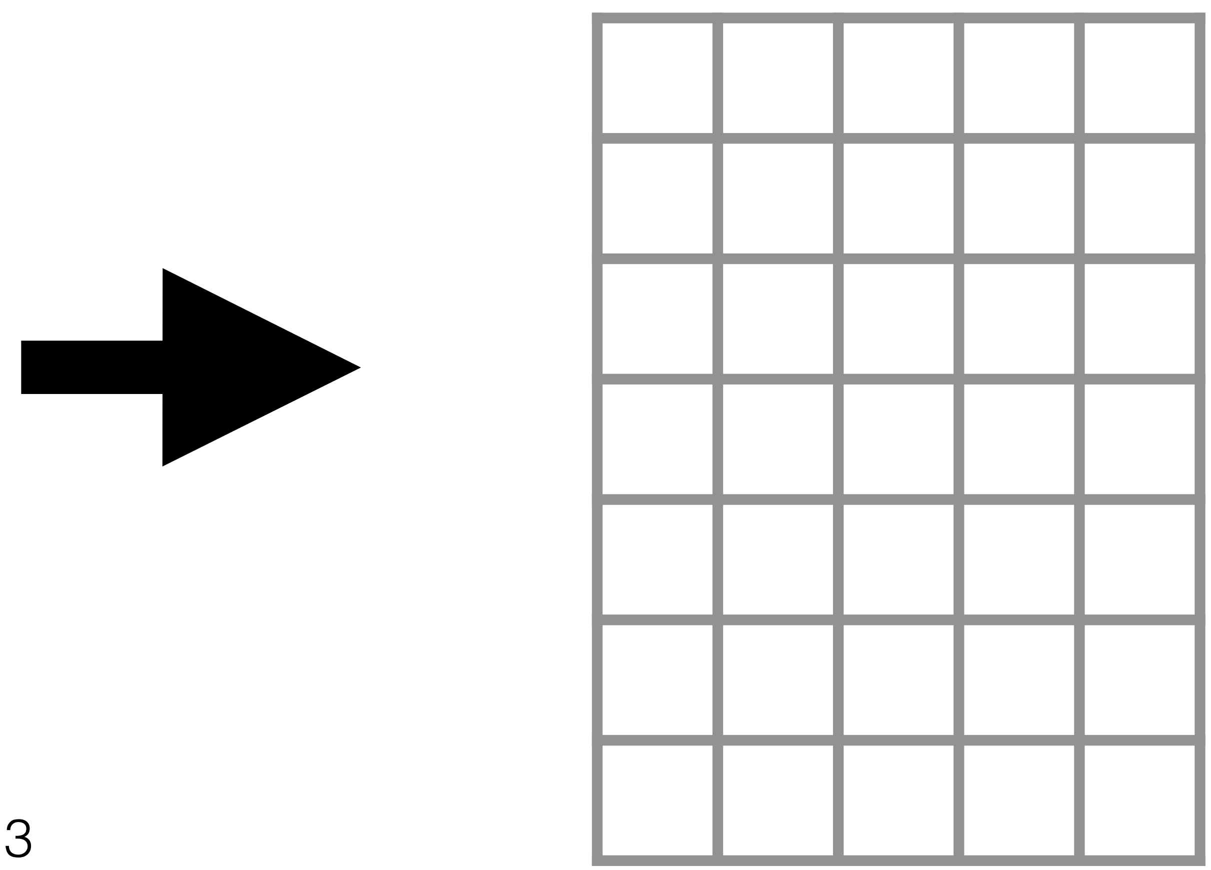

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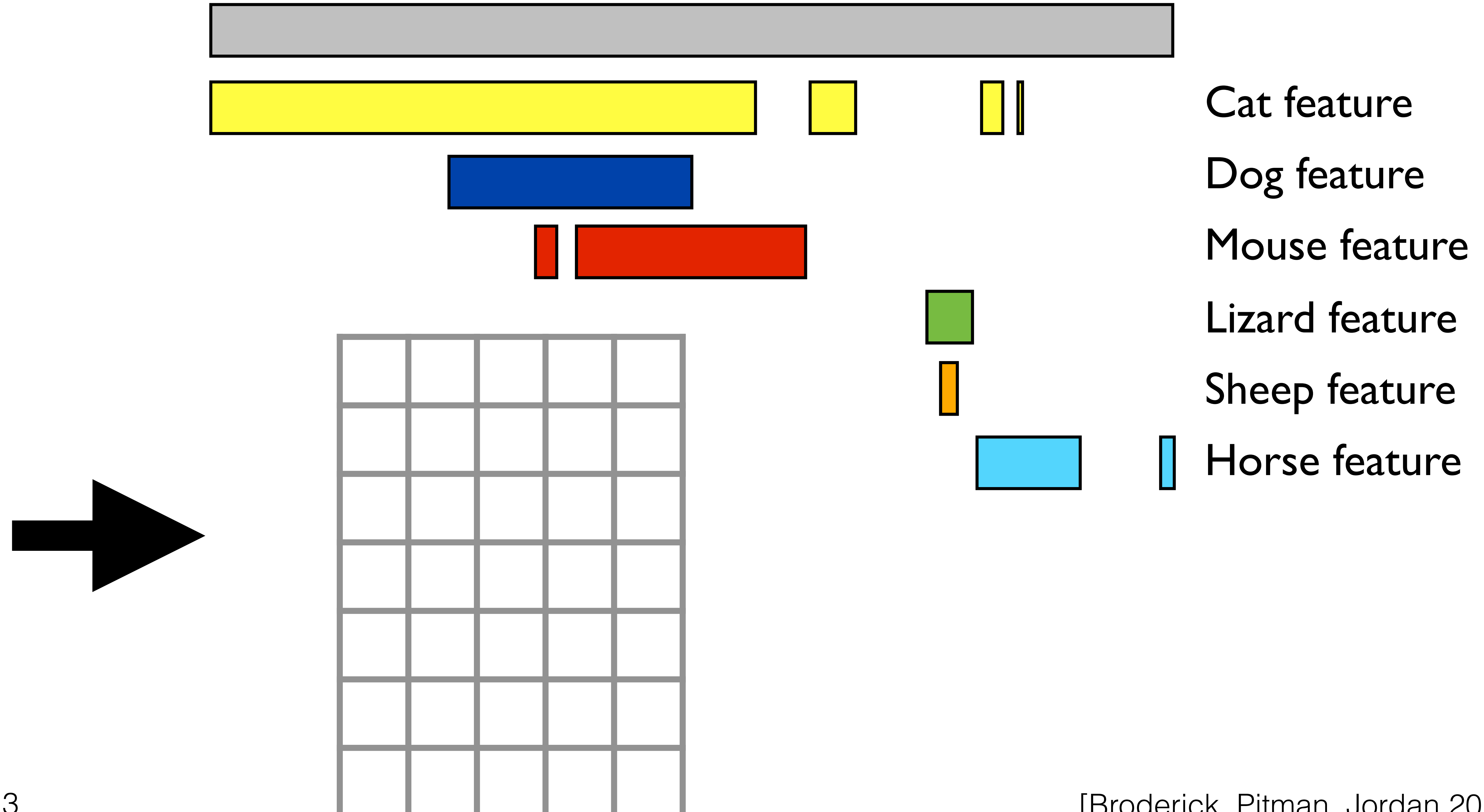
Exchangeable feature allocation: feature paintbox



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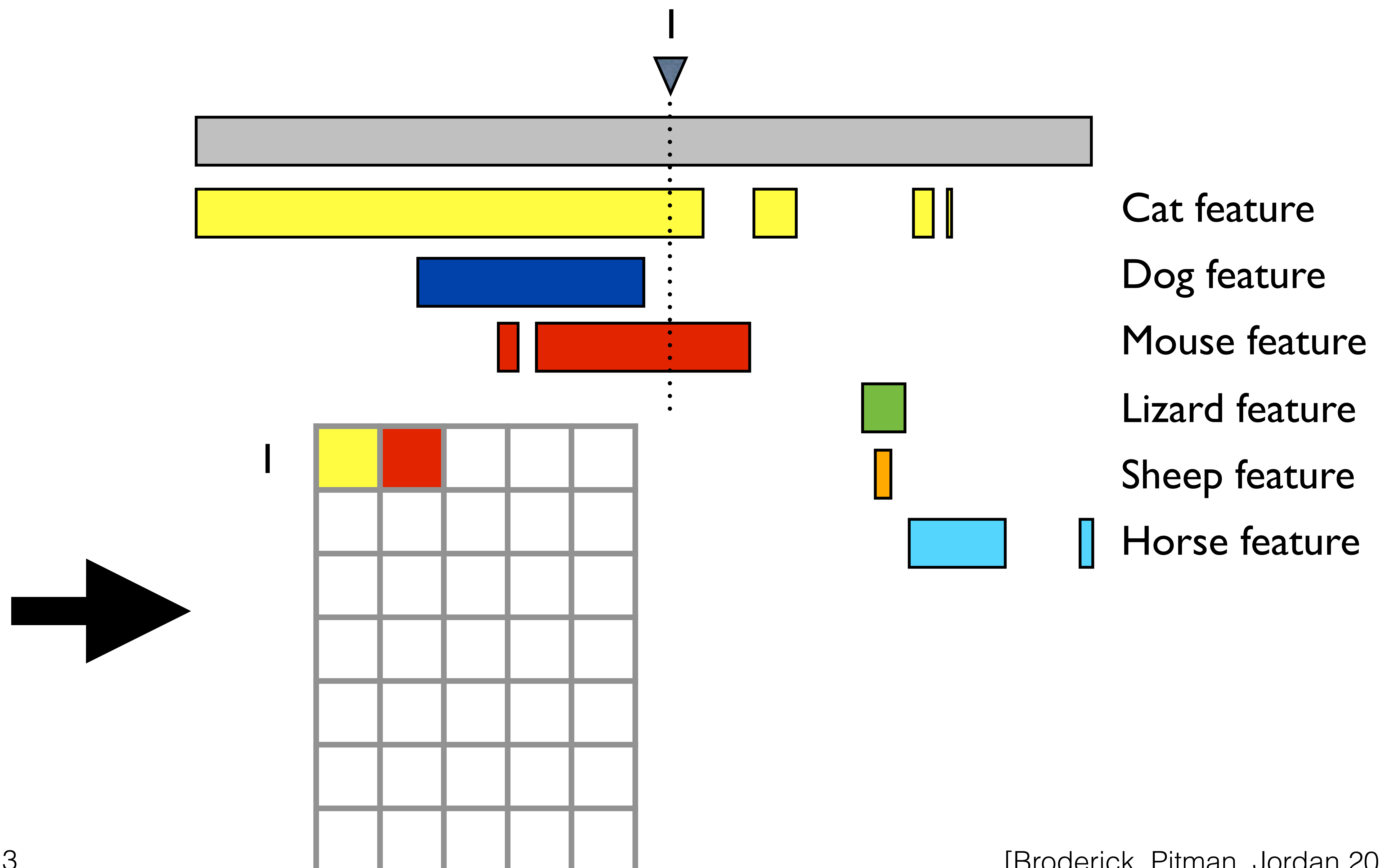

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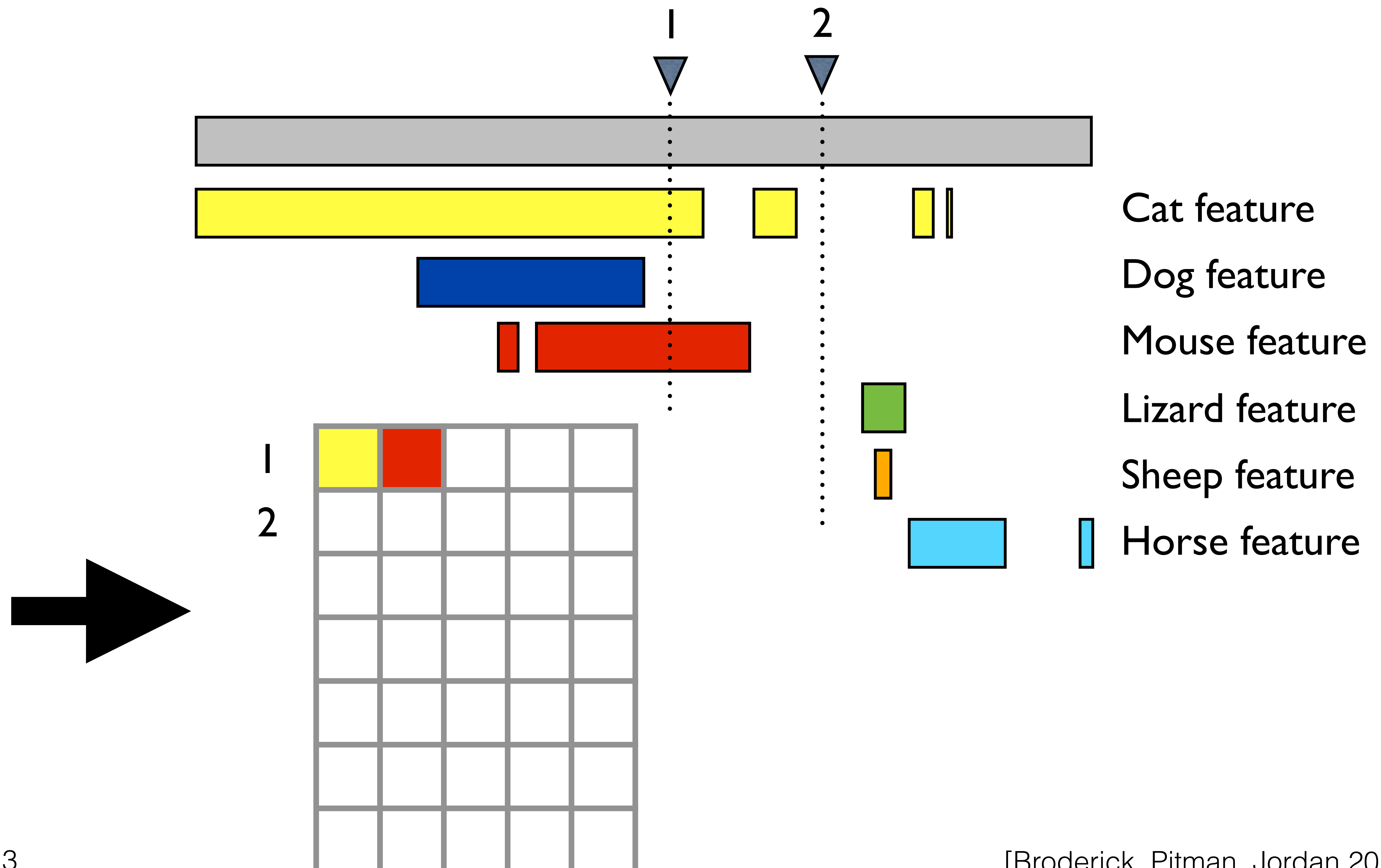

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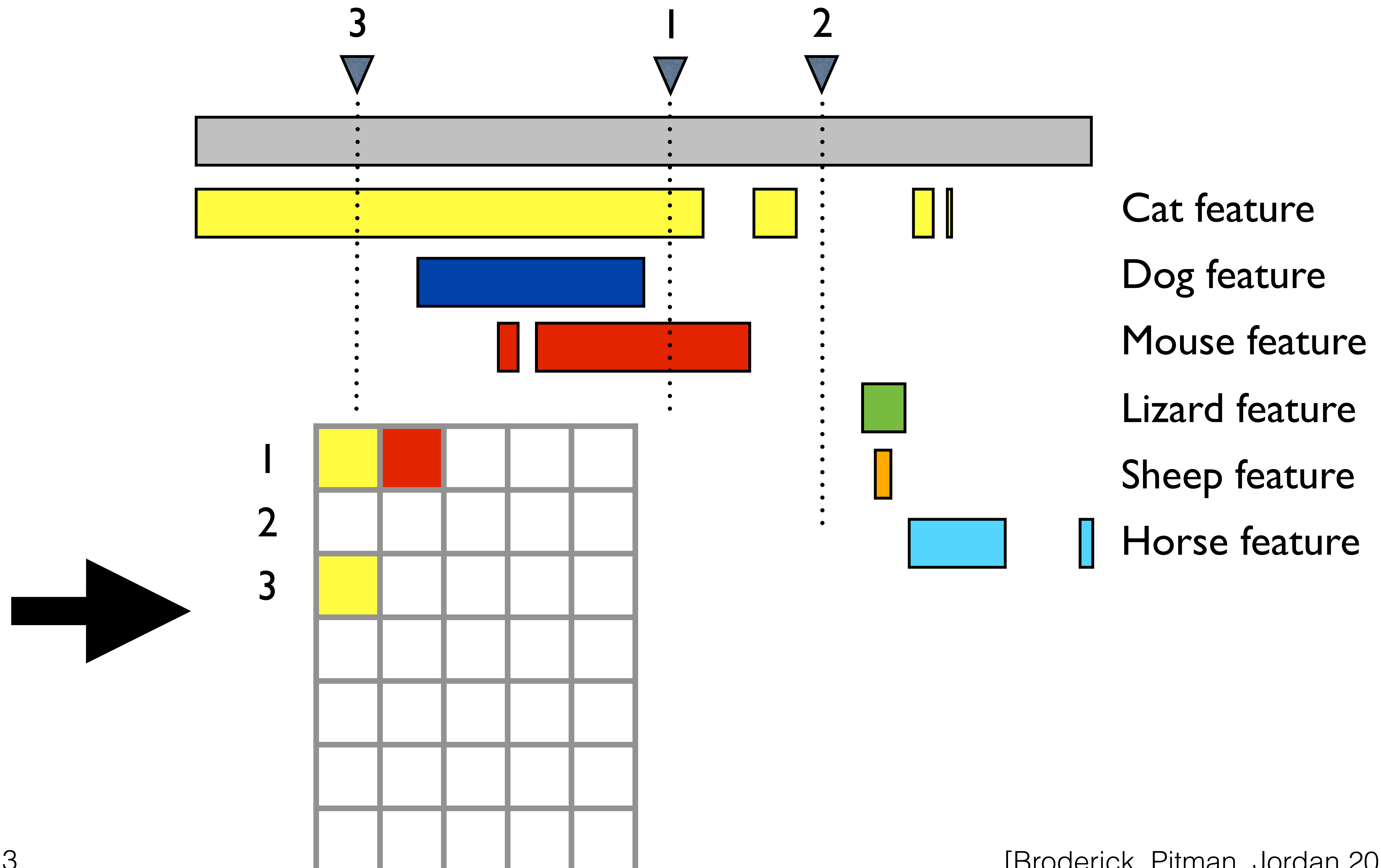

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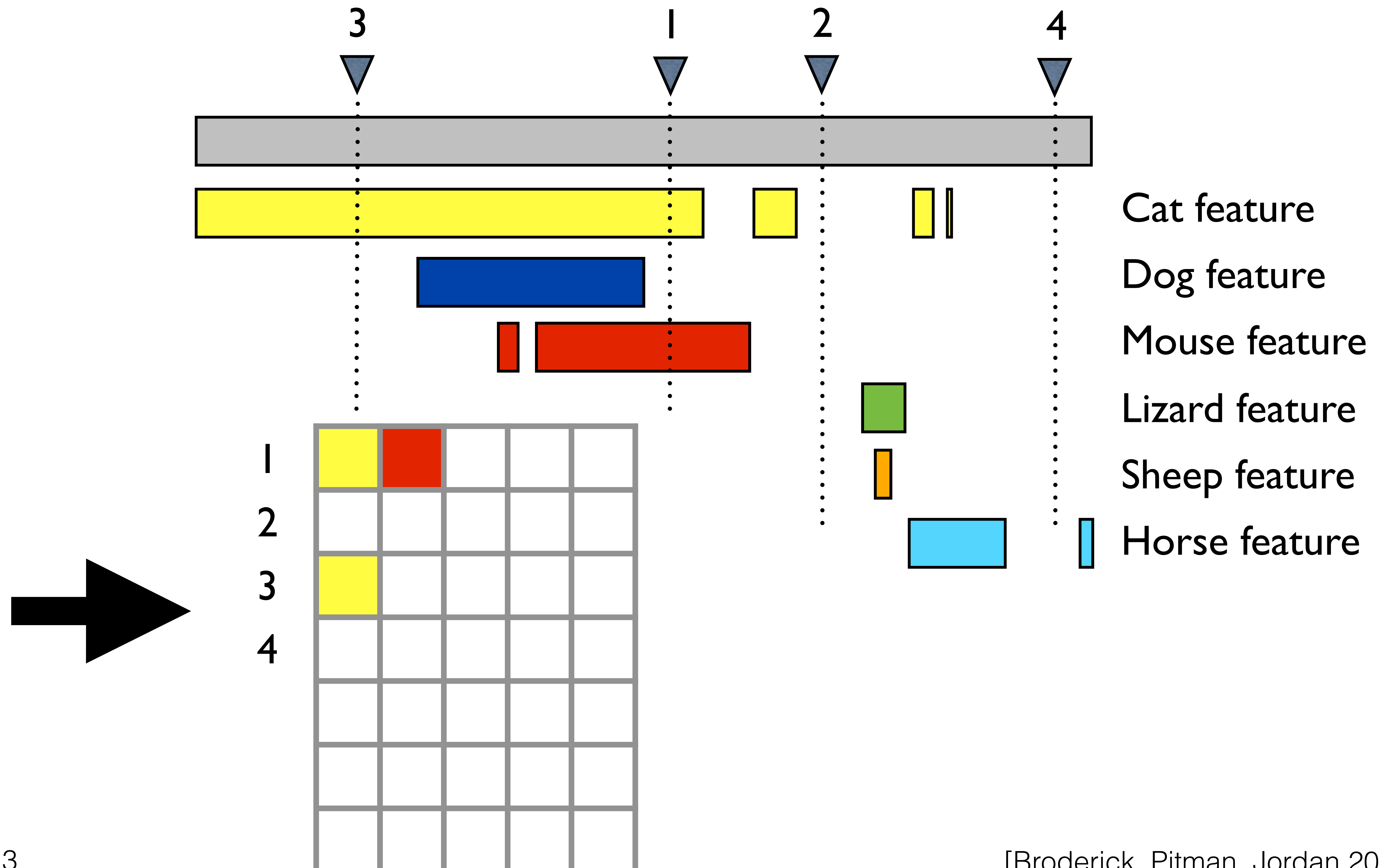

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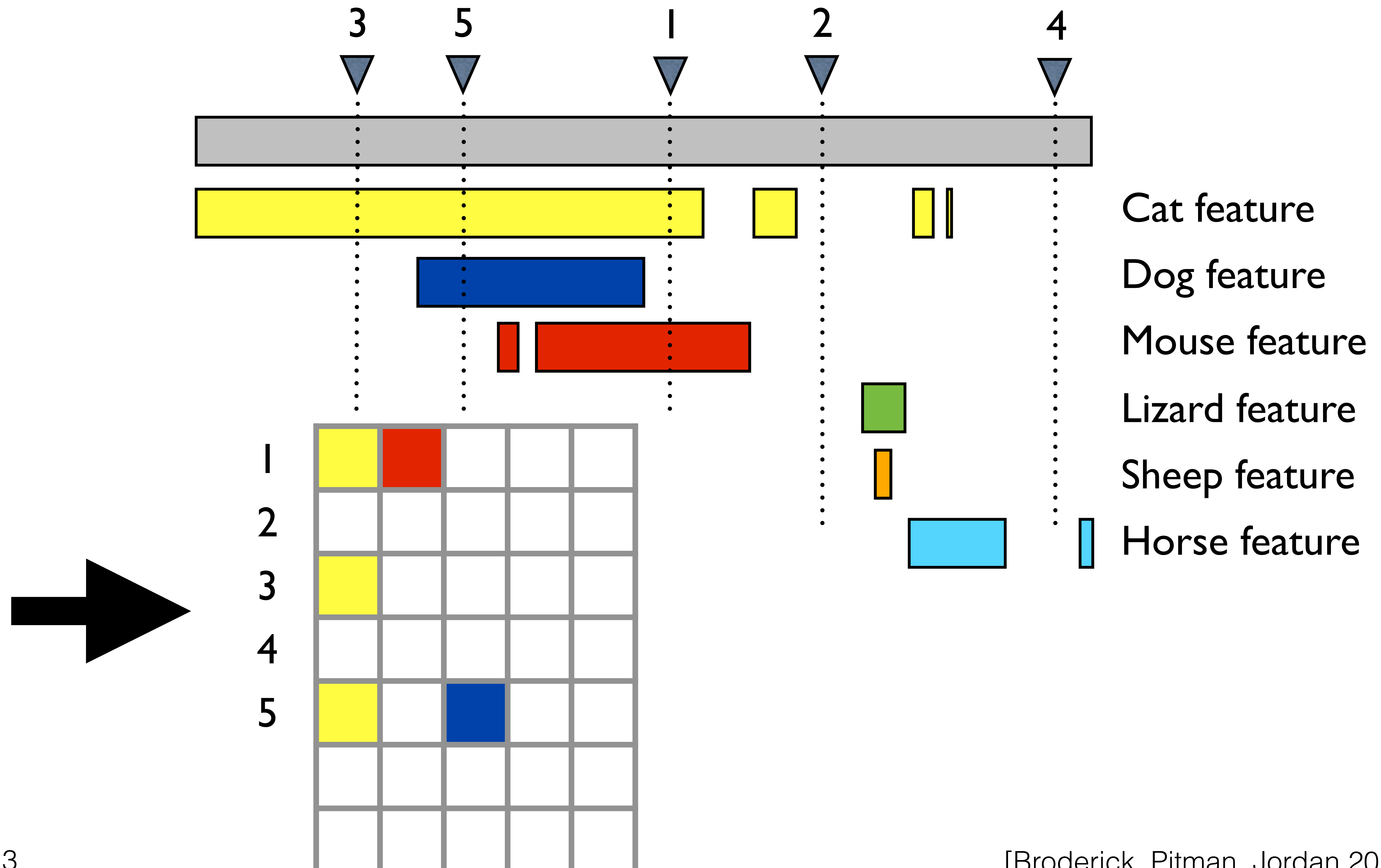

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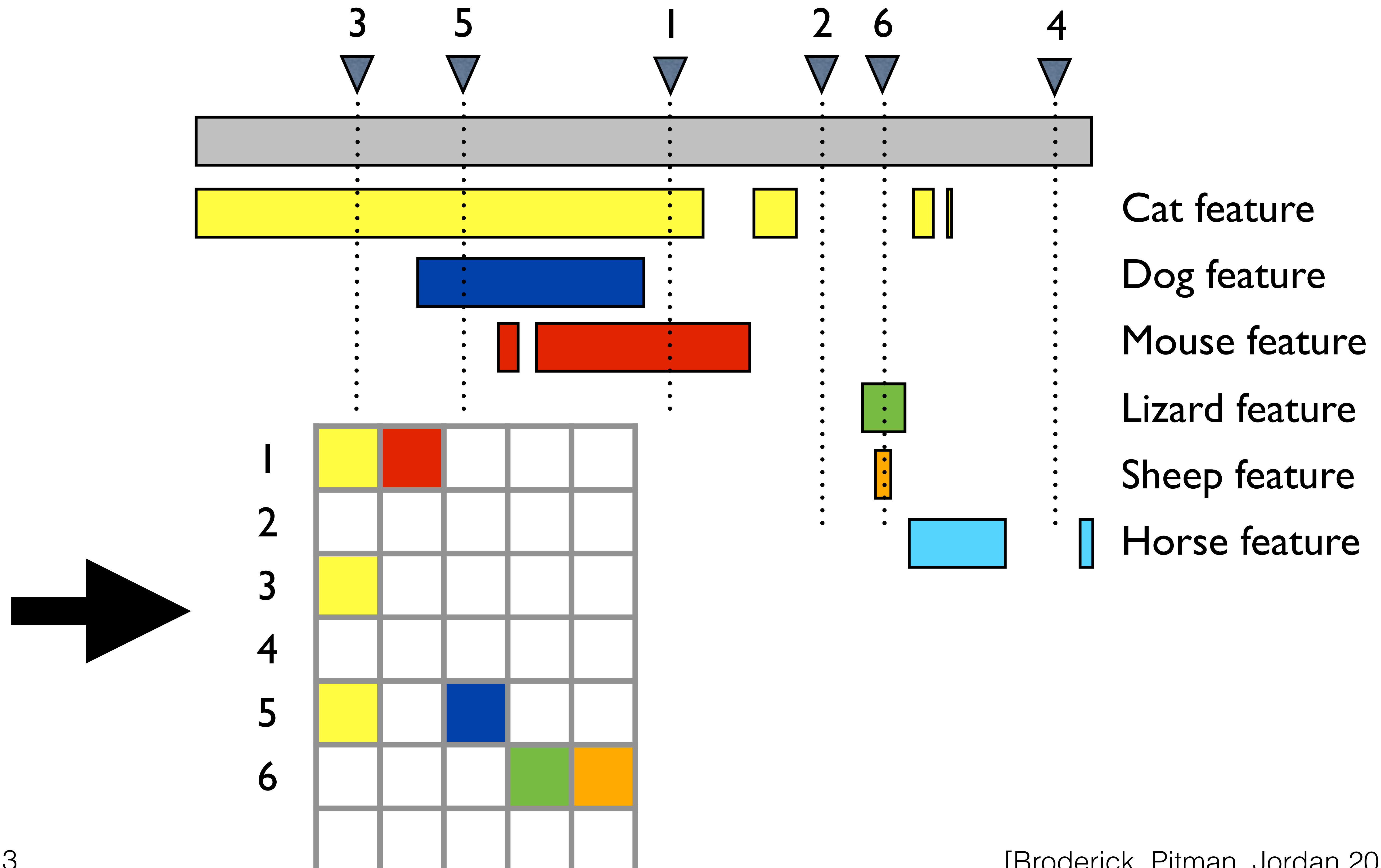

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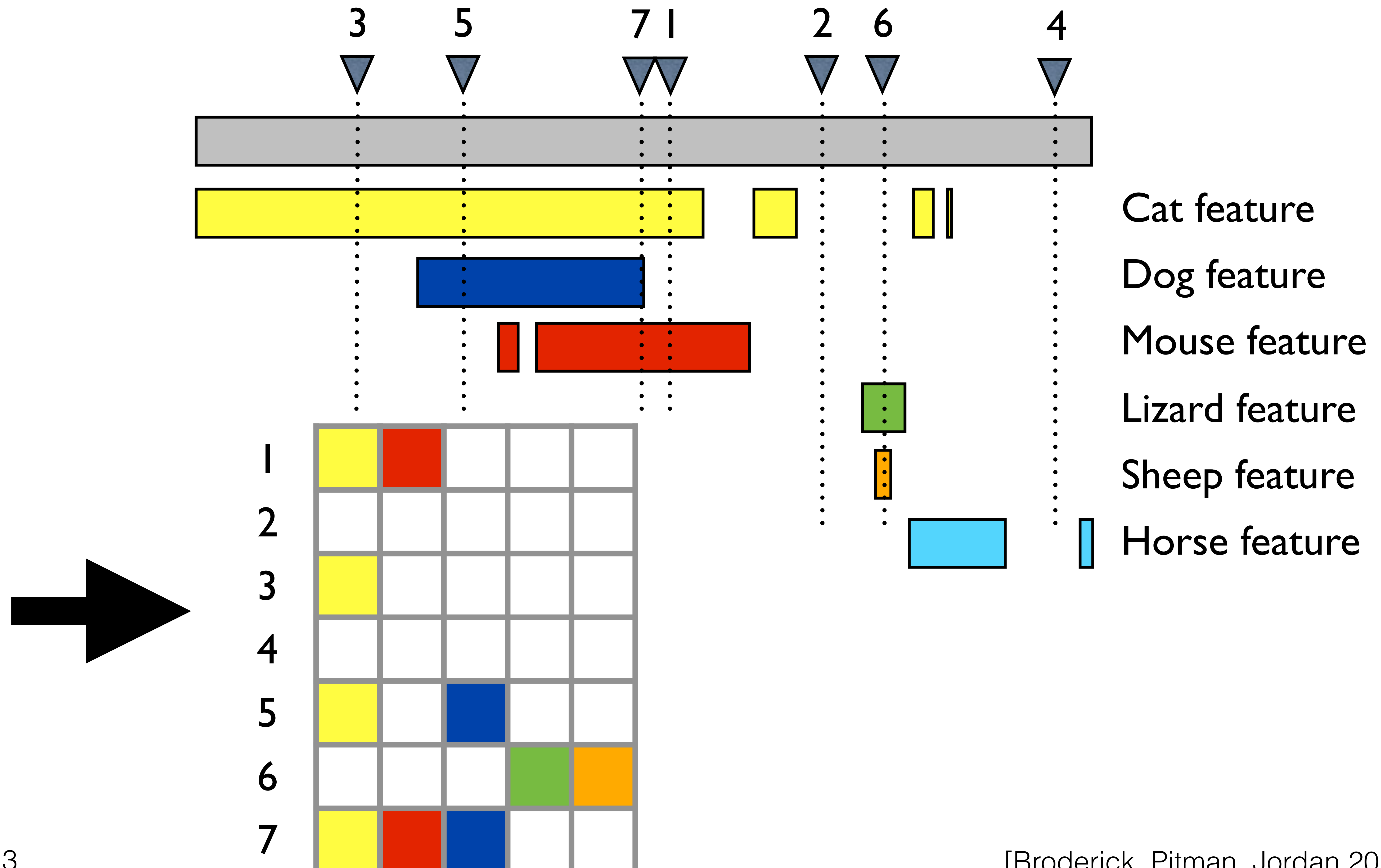

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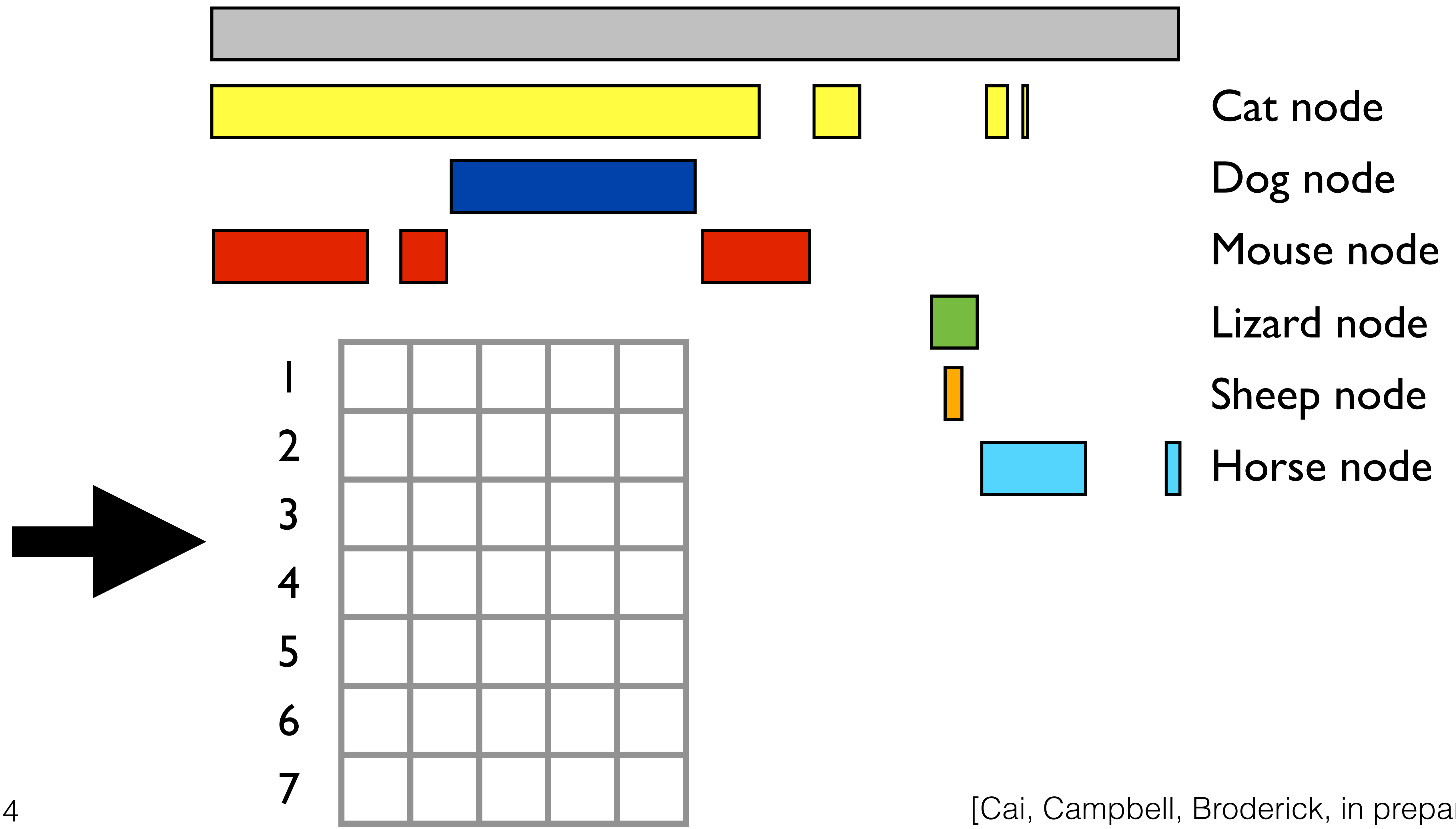
Edge-exchangeable graph



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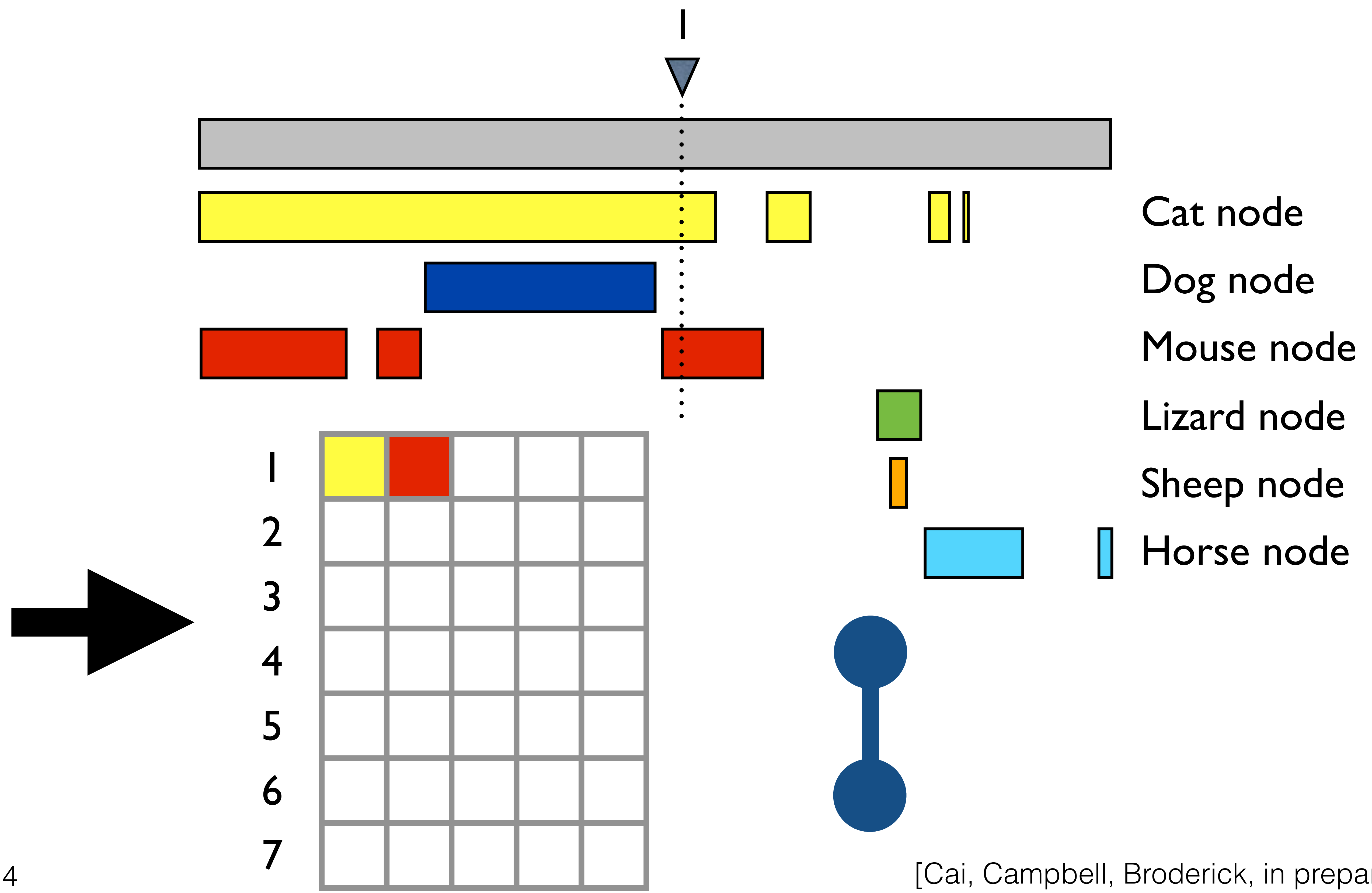

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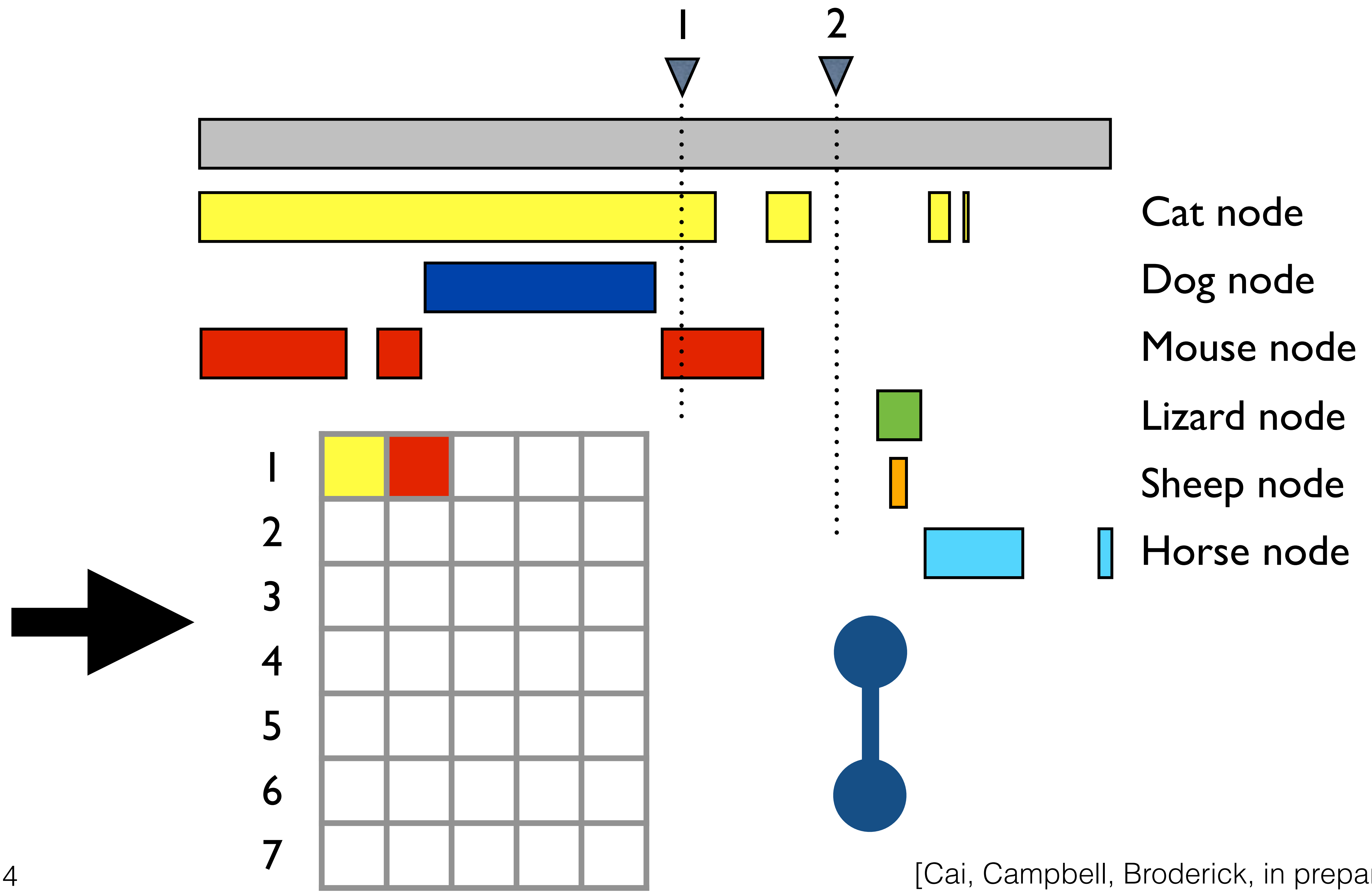

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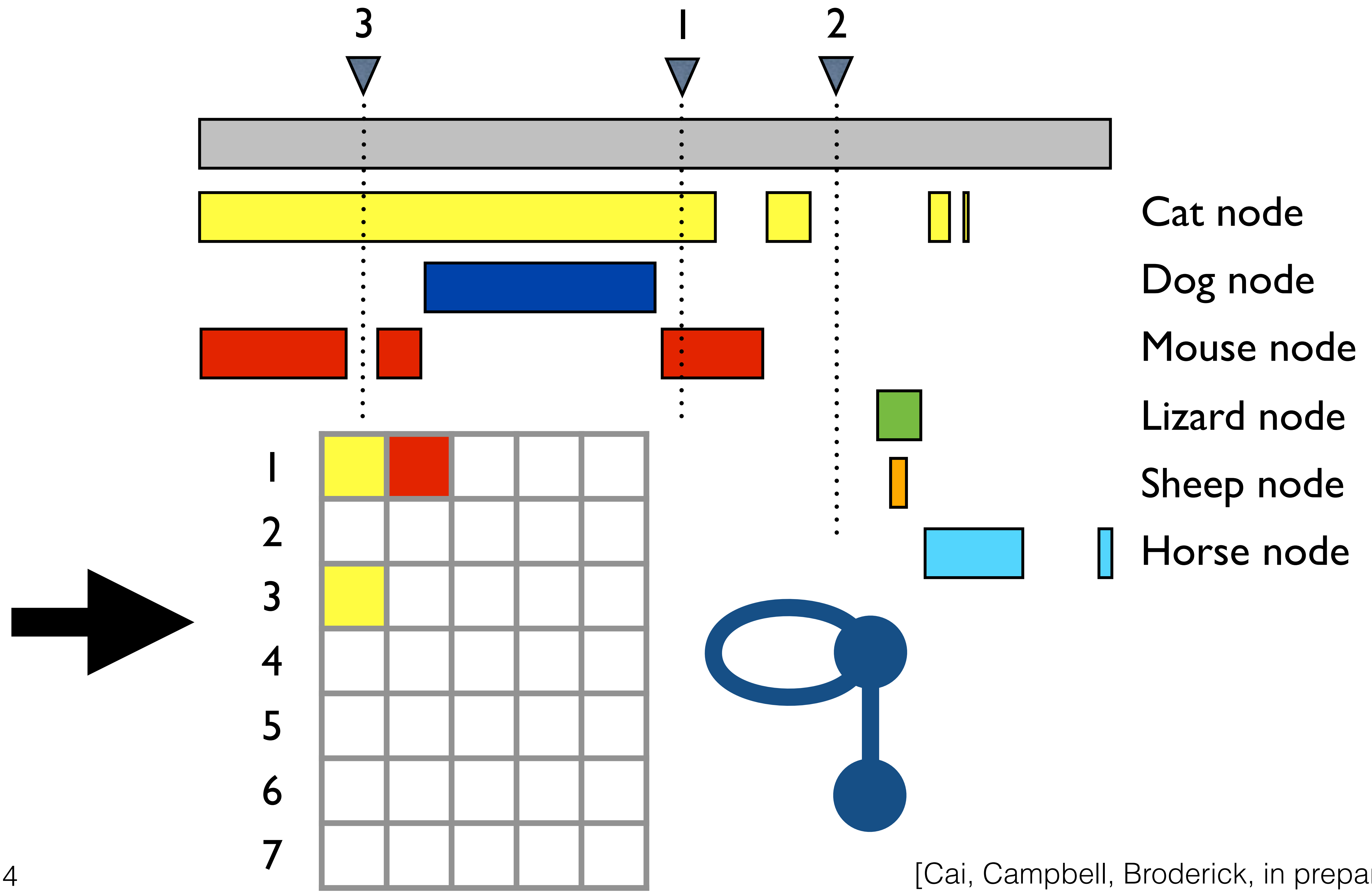

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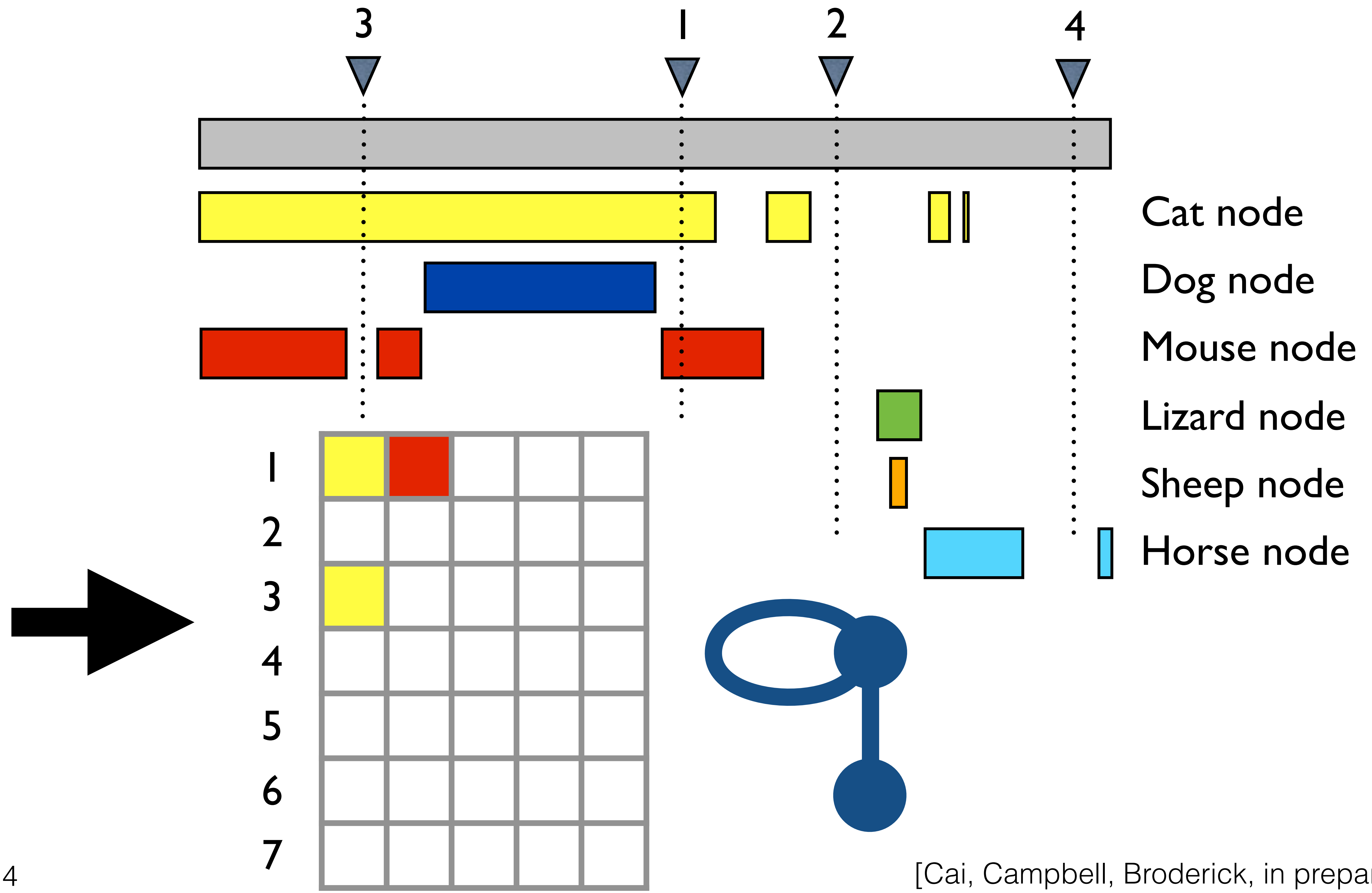

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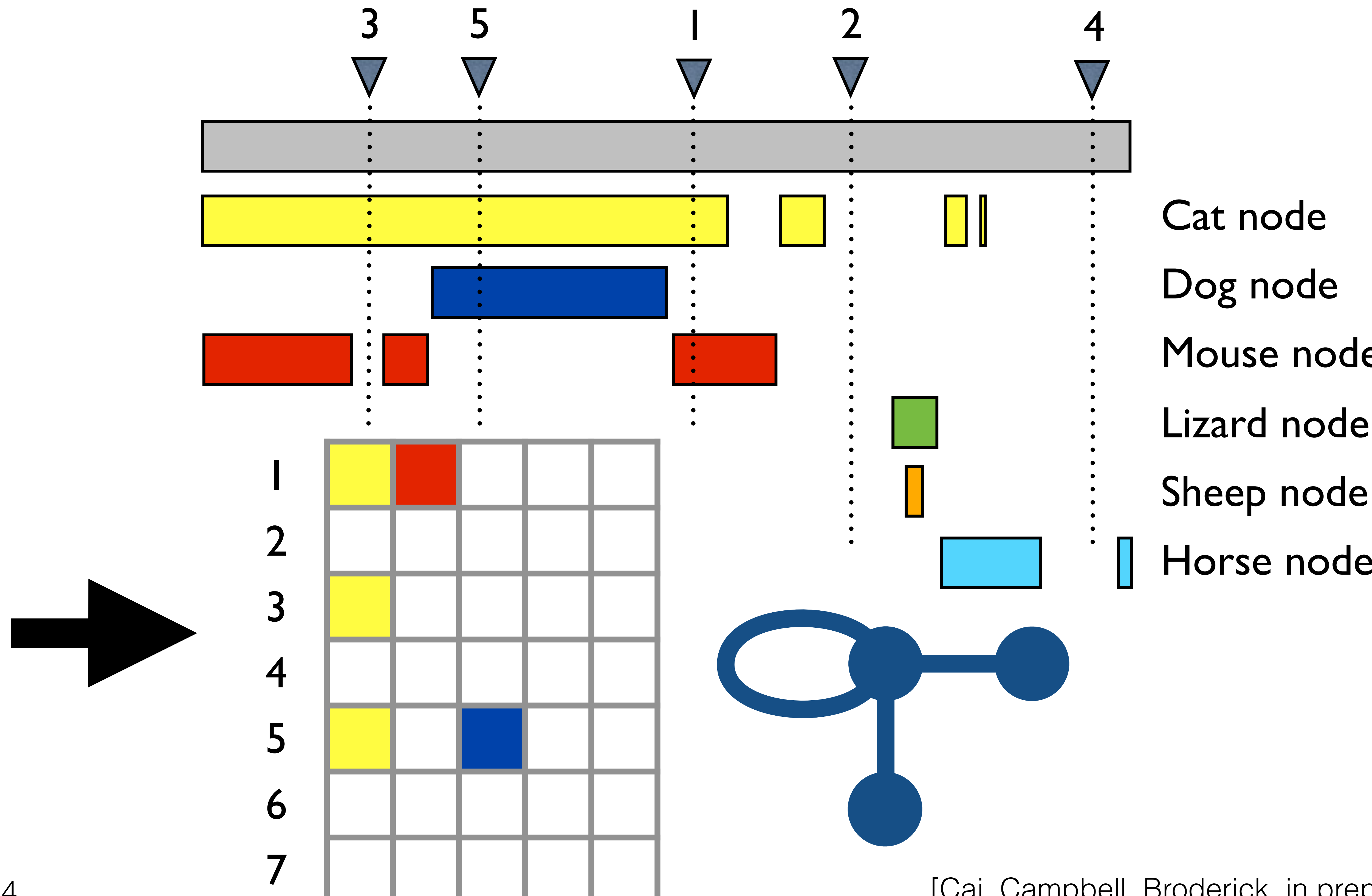

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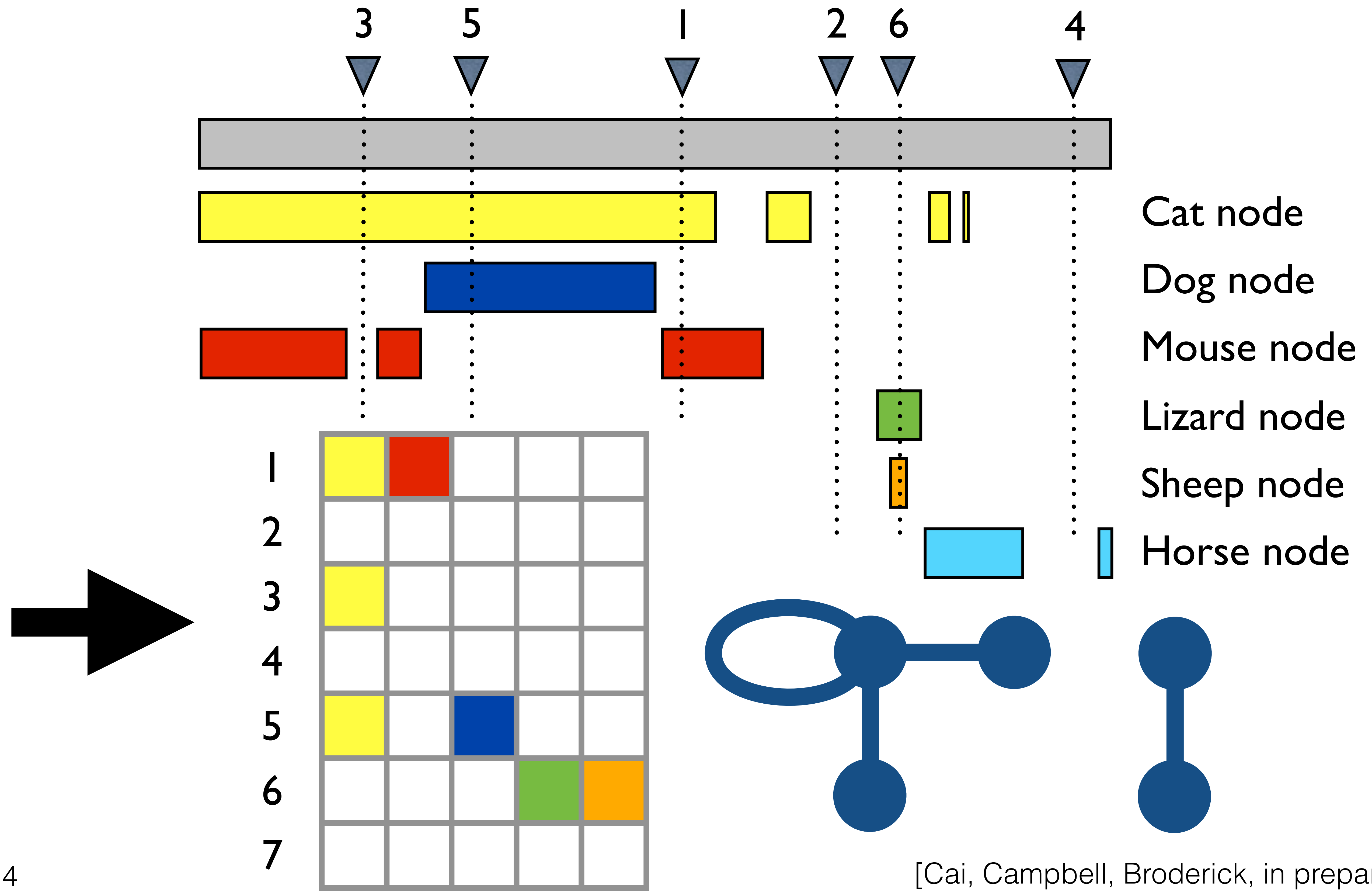

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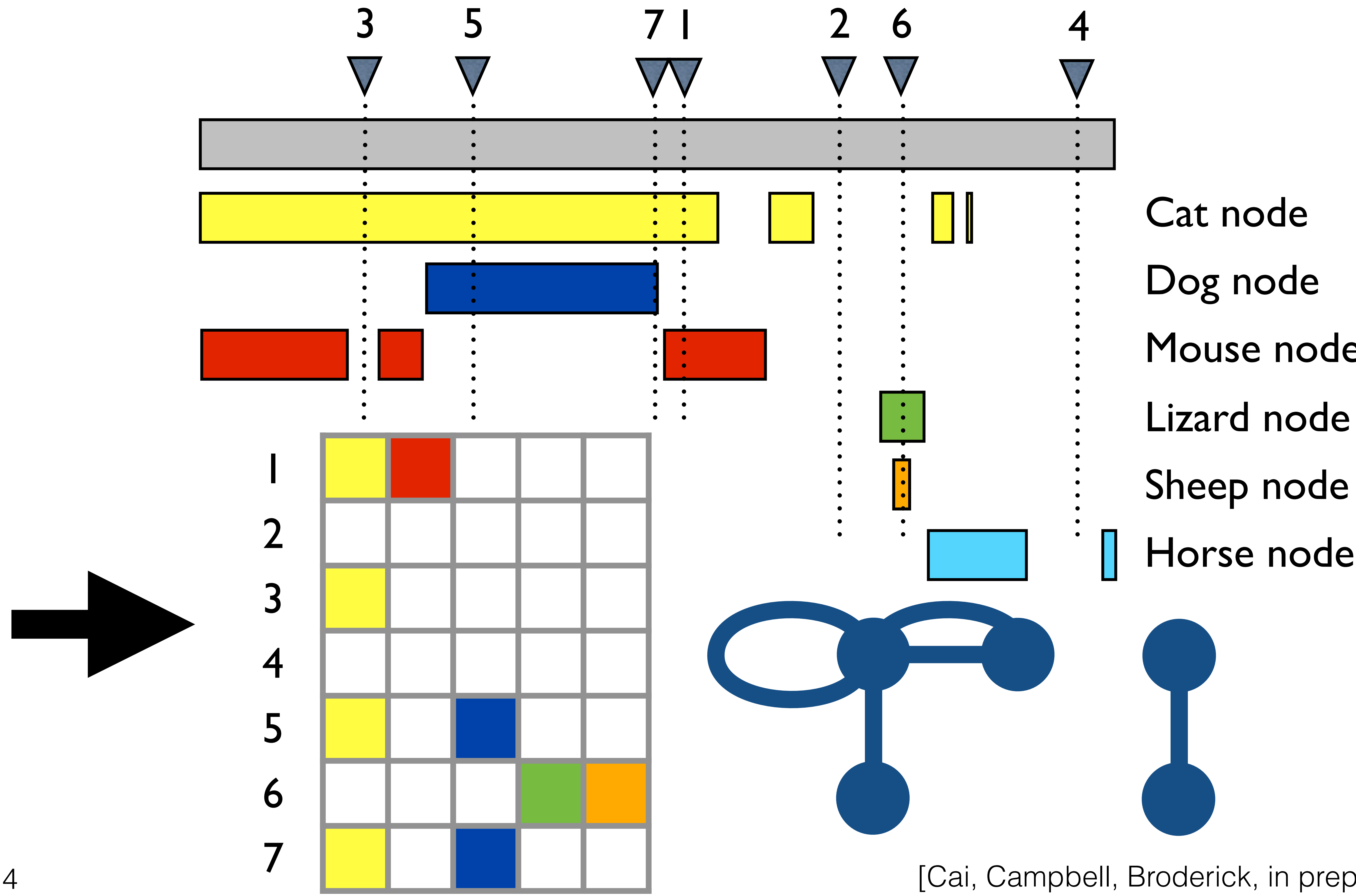

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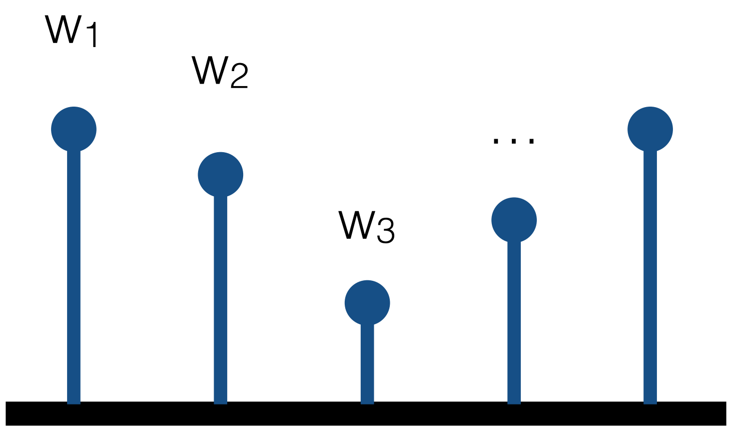
Allows sparse graphs

1. Draw discrete random measure $\sum_{j=1}^{\infty} w_j \delta_{\theta_j}$



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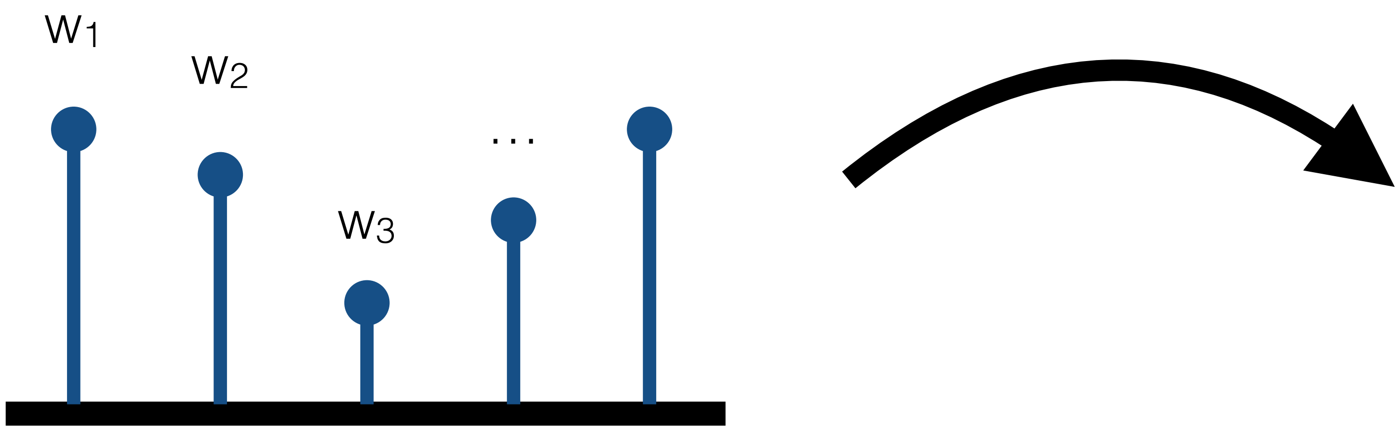

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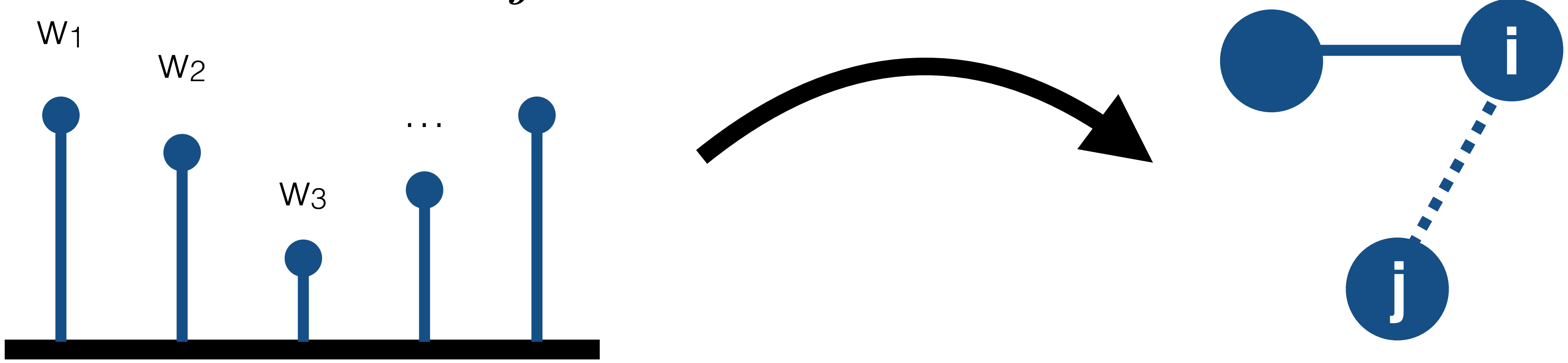

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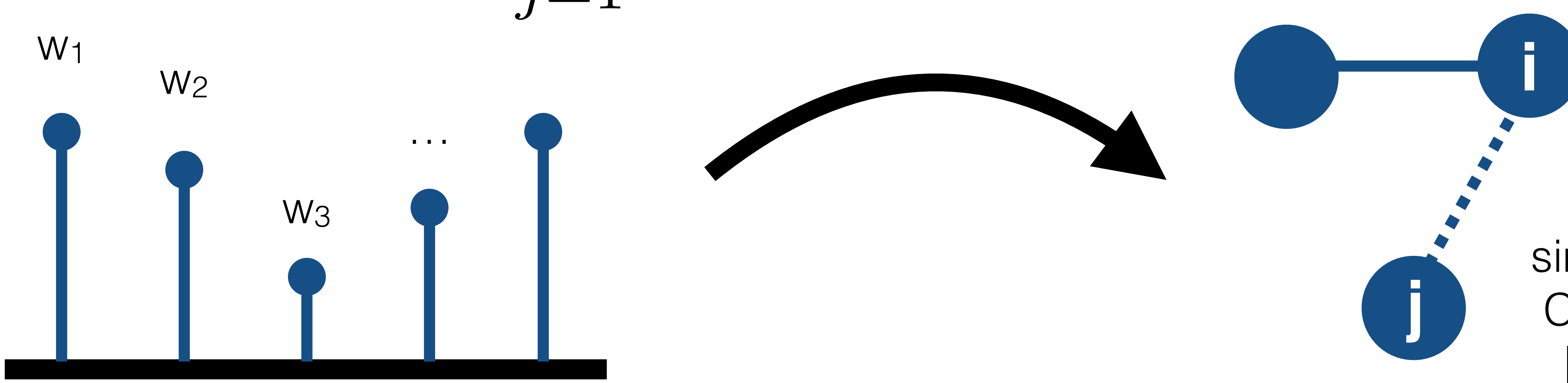

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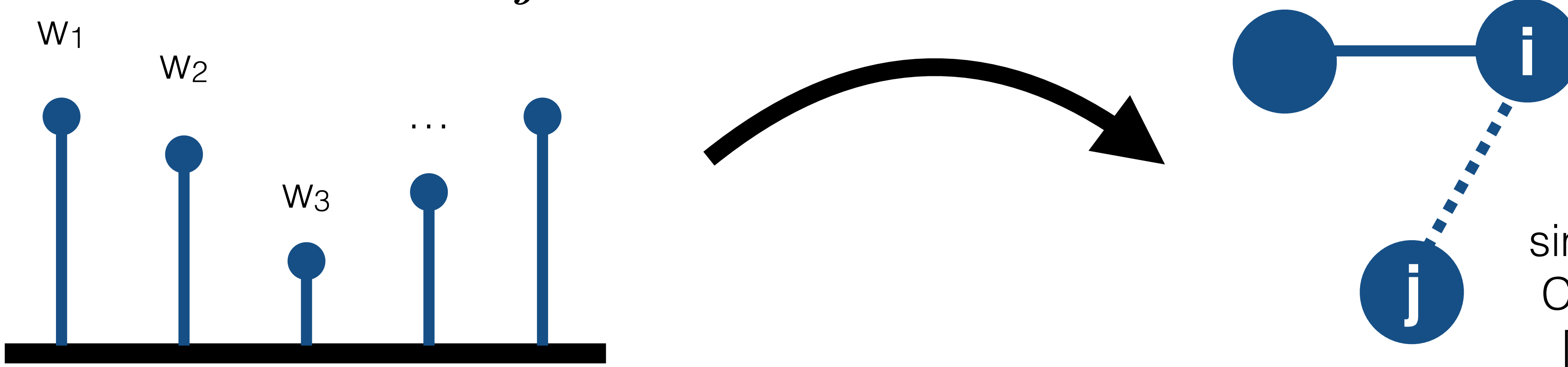

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- "Graph frequency model"

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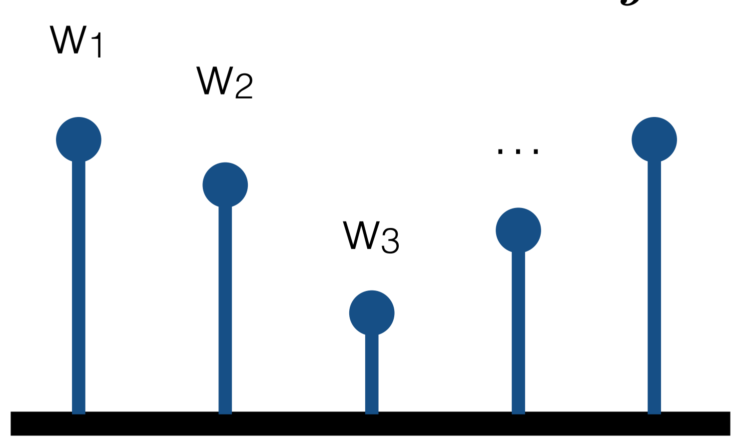

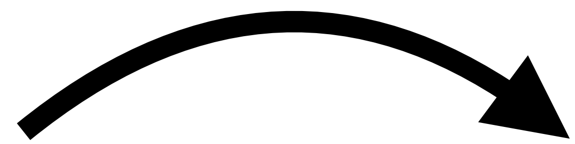

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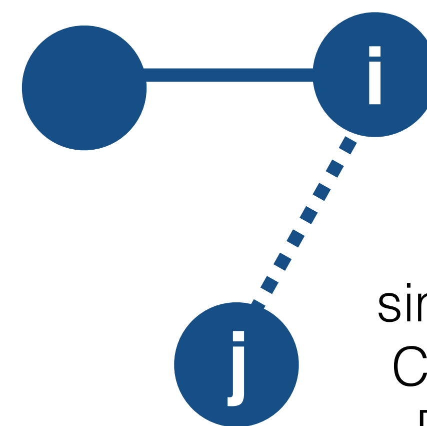

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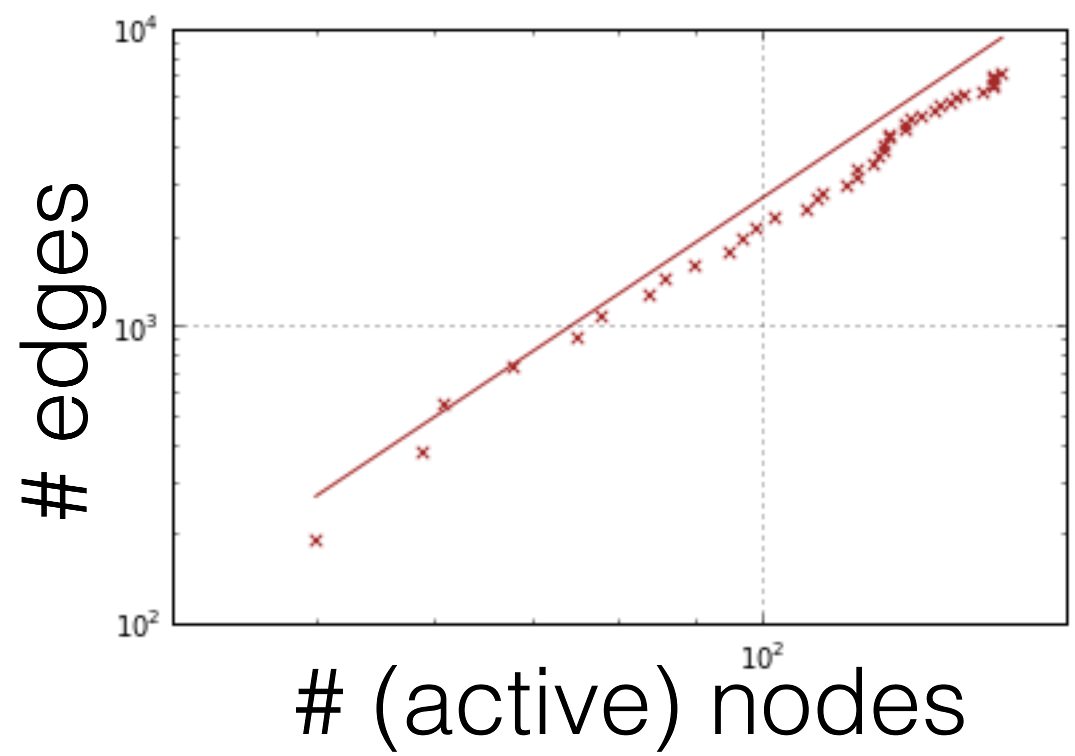

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- Prove an EPPF-style representation
- Truncation approximations

[Campbell, Huggins, How, Broderick 2016]

References

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Cai D, Broderick T. Completely random measures for modeling power laws in sparse graphs. *NIPS 2015 Workshop on Networks in the Social and Information Sciences*. 2015.

Campbell T*, Huggins JH*, How J, and Broderick T. Truncated random measures. Under review. Preprint on arXiv:1603.00861