Causal Inference in the Presence of Networks

Randomization and Observation

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Graph Exploitation Symposium,

A casual stroll through causal inference

- Neyman, 1923, Rubin, 1974, etc.
- n units are potentially assigned to treatments $(Z_1, \ldots, Z_n)$.
- The potential outcome of unit i is given by $Y_i(Z_1, \ldots, Z_n)$.
- Standard assumption: $(Y_i(0), Y_i(1))$ are the PO of unit i.
- Frequently interested in the average treatment effect (ATE):

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- Networks make the above hard! Need:
  - ⋆ Randomization schemes to control interference and homophily.
  - ⋆ Matching methods for observational studies with networks.
  - ⋆ Applications: disease prevalence, social development, online advertising, business transactions.

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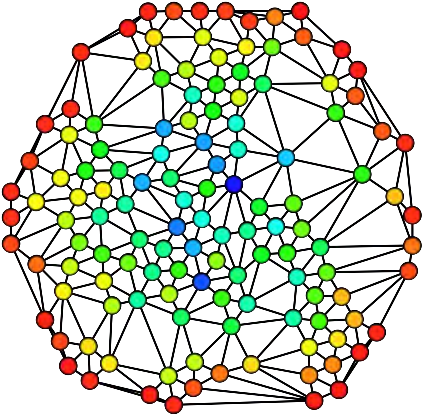
  - Total node: herd immunity.

Some context: Facebook



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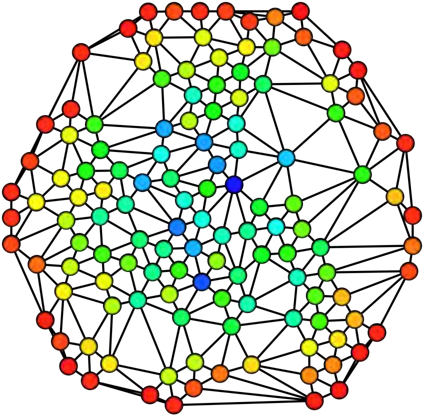

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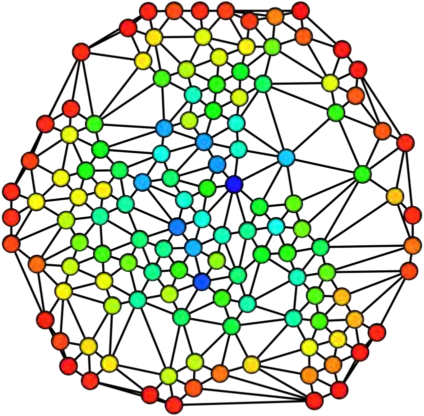

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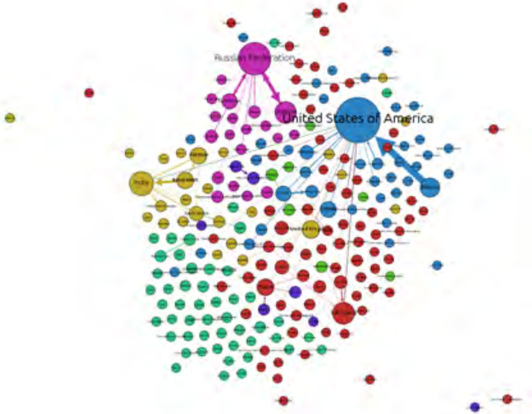

- Want to know how regime change affects population.
- Politicians during election years care about direct effects.

Source: http://openscience.alpine-geckos.at/courses/social-network-analyses/empirical-network-analysis/

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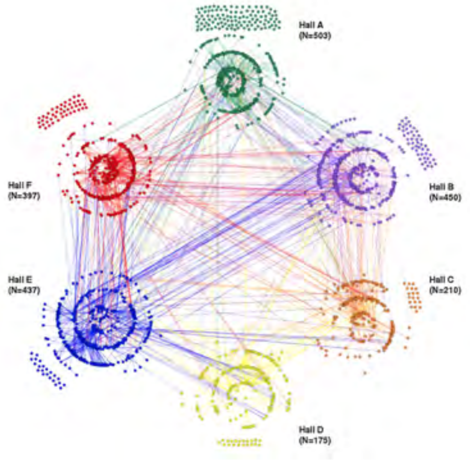

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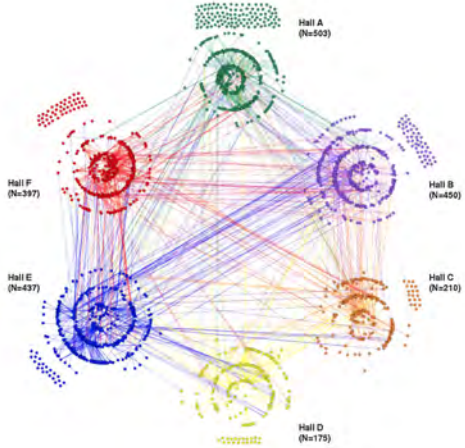

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- Interested in spread, duration of illness, etc.

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- We are interested in the direct effect!
- Simplifying assumption: interference/homophily is restricted to the neighborhood of a node.

Model and estimators

joint work with Natesh Pillai and Ravi Jagadeesan at Harvard

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- The average treatment effect is defined as $\bar{t} = \frac{1}{2n} \sum_{v \in V(G)} t_v$
- We study $|T| = n$ and the naive estimator

$$\hat{t} = \frac{1}{n} \sum_{v \in T} y_v - \sum_{v \in V(G) \setminus T} y_v$$

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Perfect assignment:

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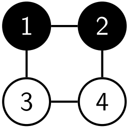

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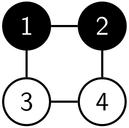

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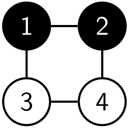

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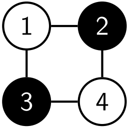

- Nodes 2,3: treated, 2 untreated neighbors.
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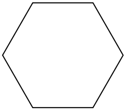

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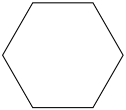

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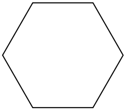

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$$\frac{1}{n} \sum_{\{w_i, w_i'\} \subseteq E(G) \cap P} \left(\frac{K_{w_i}}{d(w_i)} + \frac{K_{w_i'}}{d(w_i')} \right)$$

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$$\frac{K_1}{n} C_p + \frac{2K_2}{n} \sum_{v \in V(G)} \frac{1}{\sqrt{d(v)}}$$

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  - $C_P = \frac{1}{d_{\max}} \sum_{\{w,w'\} \in P} |d(w) - d(w')|$
  - Average of $1/\sqrt{d(v)}$.
- The bound is:

$$\frac{K_1}{n} C_p + \frac{2K_2}{n} \sum_{v \in V(G)} \frac{1}{\sqrt{d(v)}}$$

- We can use this information to build a better partition!

What about variance?

- Simplification and complication...
- Consider $f_v(S) = f(|S|, |\mathcal{N}(v) \setminus S)$ – this is symmetric interference.
- Let $\overrightarrow{d}(v) = (|T \cap \mathcal{N}(v)|, |\mathcal{N}(v) \setminus T|)$ be the bidegree of v.
- The bound on the variance is given as a function of
  - The partition: $P = \{\{w_1, w_1'\}, \ldots, \{w_n, w_n'\}\}$
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- We can use this information to build a better partition!
- (by controlling the C_P term)

Better partitions

- Order the vertices as $V(G) = \{w_1^\star, w_1^{\star\prime}, \dots, w_n^\star, w_n^{\star\prime}\}$ such that
$$d(w_1^\star) \geq d(w_1^{\star\prime}) \geq \cdots \geq d(w_n^\star) \geq d(w_n^{\star\prime})$$
- Define the partition as
$$P^\star = \{\{w_1^\star, w_1^{\star\prime}\}, \dots, \{w_n^\star, w_n^{\star\prime}\}\}$$
- By definition: $C_p \leq 1$.
- The bias and L^2 norm are bounded by
$$\frac{2K_2}{d_{\min}} \text{ and } \frac{K_1}{n} + \frac{2K_2}{\sqrt{d_{\min}}}$$
- In a dense graph we have $n \to \infty$ implies $d_{\min} \to \infty$.
- So MSE goes to zero!

What about sparse graphs?

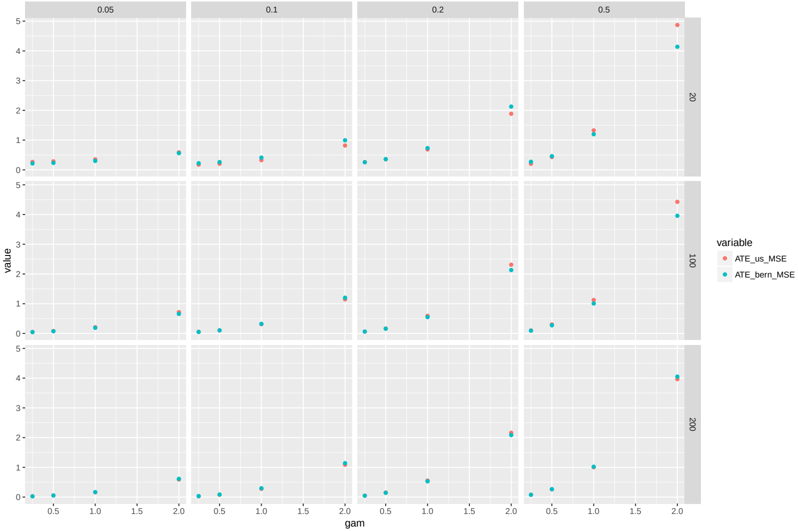
We need a little more math to get bounds that still go to zero

What about fancier interference?

Everything holds for $f_v = f_{\text{type}(v)}$(mutatis mutandis)

Small simulation

Linear interference, Erdos-Renyi graph



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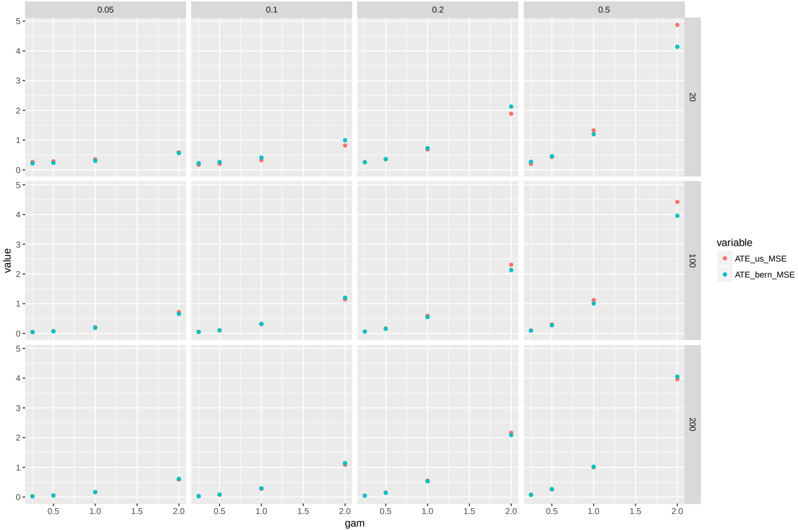
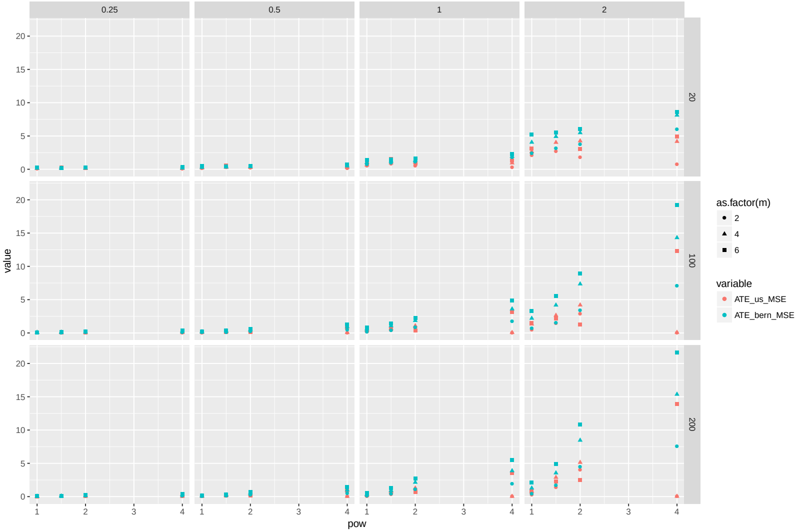


ER graphs have lots of nodes with the same degree...

Small simulation

Linear interference, Preferential-Attachment graph



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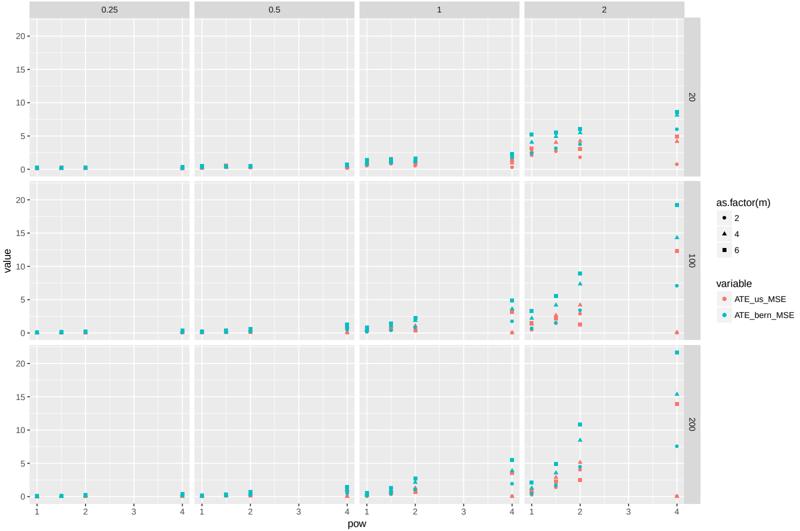
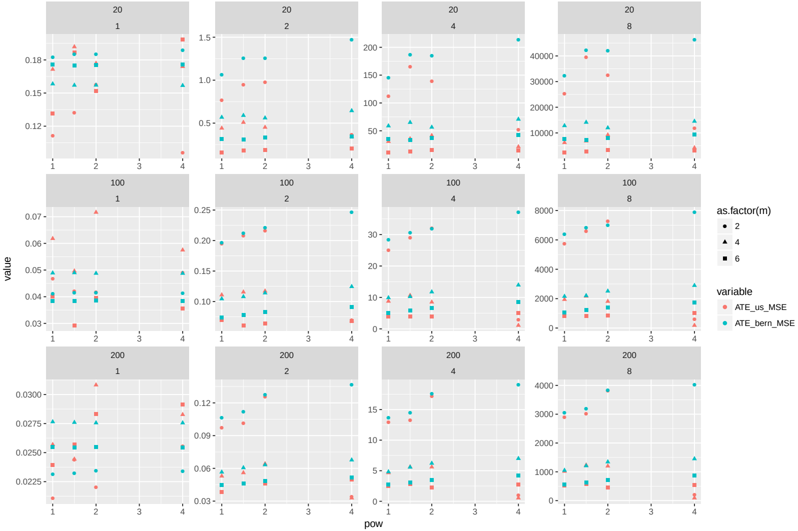


Lots of degree heterogeneity...

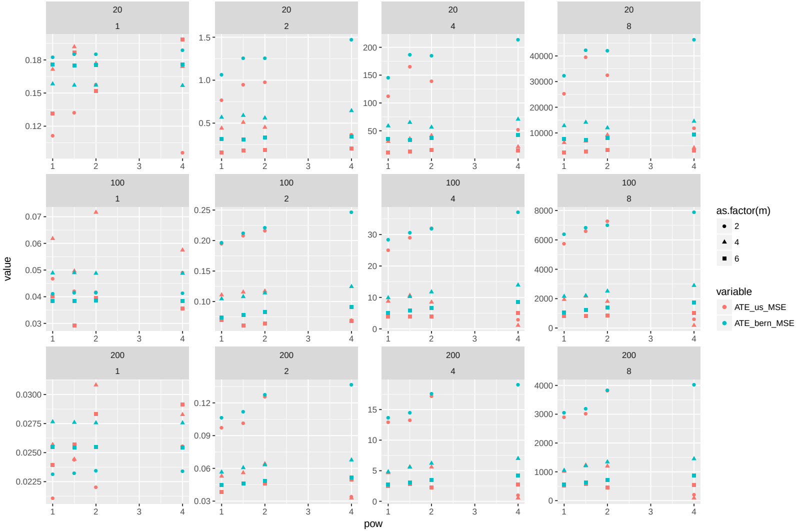
Small simulation

Generic interference, Preferential-Attachment graph



Small simulation

Generic interference, Preferential-Attachment graph



Lots of degree heterogeneity... and possibly huge interference

Natural extension to homophily

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- Given such a collection of types Π and a bound on the variability of individuals inside each type σ^2 we have similar looking bounds on bias and MSE.
- Need information on $(x_v - \sum_{v \in \pi} x_v))^2$.
- Interesting conclusion: if we can identify these "types" well then a new cluster-randomized-design is reasonable for estimating the direct effect: treat half of every "type".

Interference/homophily makes network experiments hard...

Interference/homophily makes network experiments hard... and some randomization schemes are better than others.

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Observational studies with network data are hard even without any formal interference or homophily...

Observational studies and entangled treatments

joint work with Panos Toulis at Chicago Booth and Edo Airoldi at Harvard

- Most work concentrates on questions of interference of outcomes.
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- Most work concentrates on questions of interference of outcomes.
- Lets take a step back from that — what if the treatments are entangled?

- Treatment: number of new friends in an online game.
- Treatment: popularity measure of a website due to new links.
- Treatment: number of new professional connections.
- Treatment: number of new people in a working group.

Toy example

pre-treatment network

There are two individuals and the pre-treatment period network G^- is disconnected:

Toy example

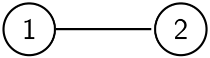
pre-treatment network

There are two individuals and the pre-treatment period network G^{-} is disconnected:

post-treatment network

No one is treated, that is $Y_1(0,0)$ and $Y_2(0,0)$ are observed.

OR



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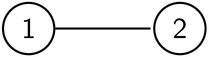

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The treatment is "number of new friends" which is an edge count — and we can't observe one person with an edge and one without.

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- Still in an observational framework so need to understand how to perform matching/weighting.
- Many estimands of interest:

$$\tau_m = E(Y_i(m+1)) - E(Y_i(m)).$$

So what goes wrong?

- Classical methods assume that no interference means we can write $Y_i(Z_i)$ and will in turn model the following propensity:

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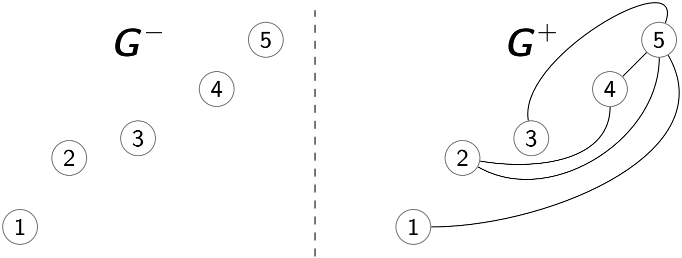
$$P(Z_i = k | X, G^-) = \int_{f_i(G^-, G^+) = k} p(G^+ | G^-, X) d\mu(G^+)$$

- This accounts for the uncertainty in the treatment due to the network evolving from G^- to G^+.

Numerical example

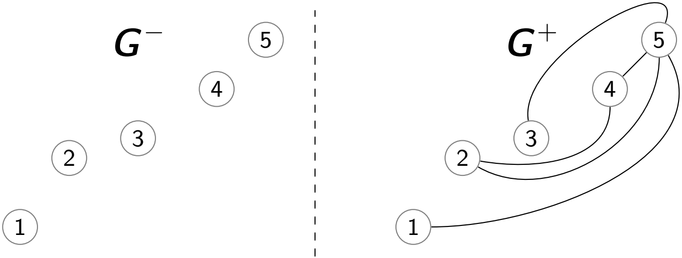


The network G^- is empty and G^+ has independent edges, each of which has probability

$$P(g_{ij}^+ = 1 | G^-, X) \propto \exp(X_i X_j + 1).$$

unit	X_i	Z_i	Y_i^{obs}
1	-5	1	0
2	-1	2	0
3	0	1	1
4	3	2	1
5	10	4	0

Numerical example



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Numerical example – results

Ignoring information about the network we fit

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	propensity score for $Z_i = \ldots$						
unit (i)	0	1	2	3	4	5	...
1	0.37	0.37	0.18	0.06	0.02	0.00	...
2	0.24	0.34	0.25	0.12	0.04	0.01	...
3	0.21	0.33	0.26	0.13	0.05	0.02	...
4	0.13	0.26	0.27	0.19	0.10	0.04	...
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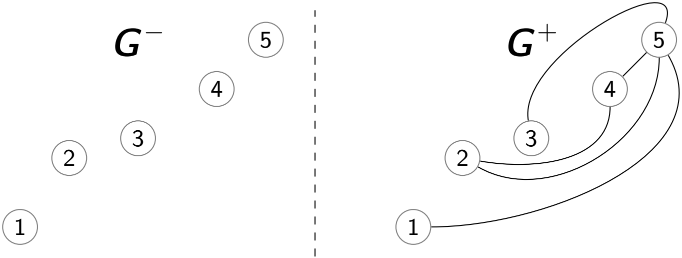
Set of units that have similar propensities to make one connection or two connections: $\mathcal{S} = \{1, 2, 3, 4\}$

Numerical example



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Numerical example – results

Using the information about the network:

	propensity for $Z_i = \dots$				
unit (i)	0	1	2	3	4
1	0.00	0.27	0.73	0.00	0.00
2	0.00	0.24	0.67	0.09	0.00
3	0.01	0.06	0.23	0.42	0.28
4	0.00	0.24	0.68	0.09	0.00
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Practical guide

- Integral is usually analytically intractable.
- Fit favorite model for $G^+|G^-, X$.
- Sample J networks from the fitted model.
- Use the samples $\{G^+_{(j)}\}$, $j = 1, \ldots, J$ to compute estimates $\hat{e}(k, X)$ of the propensity score $e(k, X)$:

$$\hat{e}(k, X) = \frac{1}{J}\sum_{j=1}^{J} \mathbb{I}\{f_i(G^-, G^+_{(j)}) = k\}$$

- Group according to estimated propensity scores.
- Compute estimates within groups and combine information across groups.

All nice in practice, but how does it work in theory?

- Matching on the correct propensity score means the assignment is ignorable.
- In that case the conditional distributions of covariates for treated and control are the same.
- What happens if you condition on the wrong propensity score?
- Let $S_c(X), S_w(X)$ be the correct and wrong propensity score given covariates X. For $\mu(s) = E(S_c|S_w = s)$ we have

$$
\begin{aligned}
E(X|Z = 1, S_w = s) - E(X|Z = 0, S_w = s) \\
= \frac{cov(S_c(X), X|S_w = s)}{\mu(s)(1 - \mu(s))}
\end{aligned}
$$

Some thoughts about moving forward

- We have a new design for experiments on networks.
  - Gives estimates of the direct effect.
  - Controls bias and MSE!
- How do we port this to observational studies?

- We develop entangled treatments in observational studies.
  - Theory for balancing of covariates.
  - Random graph connects with network analysis.
- How do we port this to randomization schemes?

- How do we do any of this fast?
- How do we communicate these ideas to practitioners?

Thank you!

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Example

- Let $V(G) = \{v_1, \ldots, v_{2k}, w_1, \ldots, w_{2k}\} = V \cup W$.

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$$\xi = \frac{\gamma(\alpha(\alpha - 1) - (2k - \alpha)\alpha)}{2k}$$

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- When we use our partition scheme we have $\alpha = k$ and so $\xi = -\frac{\gamma}{2}$ is independent of the size of the graph.
- Now letting T be uniform on all possible partitions it is clear that while $E\xi \to 0$ we have a growing variance!